A Discrete Mechanics Approach to Gait Generation on Periodically Unlevel Grounds for the Compass-type Biped Robot

Tatsuya Kai
Department of Applied Electronics
Faculty of Industrial Science and Technology
Tokyo University of Science
6-3-1 Niijuku, Katsushika-ku, Tokyo 125-8585 JAPAN
Email: kai@rs.tus.ac.jp

Takeshi Shintani
Kyocera Corporation
6 Takeda Tobadono-cho, Fushimi-ku, Kyoto 612-8501, JAPAN

***Abstract*—This paper addresses a gait generation problem for the compass-type biped robot on periodically unlevel grounds. We first derive the continuous/discrete compass-type biped robots (CCBR/DCBR) via continuous/discrete mechanics, respectively. Next, we formulate a optimal gait generation problem on periodically unlevel grounds for the DCBR as a finite dimensional nonlinear optimization problem, and show that a discrete control input can be obtained by solving the optimization problem with the sequential quadratic programming. Then, we develop a transformation method from a discrete control input into a continuous zero-order hold input based on the discrete Lagrange-d'Alembert principle. Finally, we show numerical simulations, and it turns out that our new method can generate a stable gaits on a periodically unlevel ground for the CCBR.**

I. Introduction

Numerous work on humanoid robots have been done via various approaches in the fields of robotics and control theory until now. For instance, there are the following approaches: theoretical analysis of passive walking [1], [2], [3], [4], researches associated with nonlinear dynamical theory such as Poincáre sections and limit cycles [5], [6], [7], [8], [9], [10], [11], gait pattern generation based on CPG (central pattern generation) and ZMP (zero-moment point) [12], [13], [14], [15], and self-motivating acquirement of gaits by learning theory and evolutionary computing [16], [17], [18], [19]. Especially, as one of the simplest models of humanoid robots, the compass-type biped robot has been mainly studied by a lot of researchers. In general, it is quite difficult to realize stable gaits for humanoid robots in terms of nonlinear control problems, and hence there are still a lot of problems left to solve.

In almost every work on humanoid robots, models derived by normal continuous-time mechanics are used. On the other hand, *discrete mechanics*, which is a new discretizing tool for nonlinear mechanical systems and is derived by discretization of basic principles and equations of classical mechanics, has been focused on [20], [21], [22], [23], [24], [25]. a discrete model (the discrete Euler-Lagrange equations) in discrete mechanics has some interesting characteristics; (i) less numerical error in comparison with other numerical solutions such as Euler method and Runge-Kutta method, (ii) it can describe energies for both conservative and dissipative systems with less errors, (iii) some laws of physics such as Noether's theorem are satisfied. (iv) simulations can be performed for large sampling times. Hence, discrete mechanics has a possibility of analysis and controller synthesis with high compatibility with computers.

We have focused on discrete mechanics and considered its applications to control theory. In [26], [27], [28], we applied discrete mechanics to control problems for the cart-pendulum system, and confirmed the application potentiality to control theory. Moreover, in [29], [30], [31], [32], we have considered a gait generation problem for the compass-type biped robot and confirmed that the proposed method can generate stable gaits on flats and slopes. However, the method cannot be applied to gait generation problems on more complex grounds.

Therefore, this paper aims at gait generation for the compass-type biped robot on periodically unlevel grounds which are more complex than flats and slopes from the standpoint of discrete mechanics. This paper is organized as follows. In Section II, a brief summary on discrete mechanics is presented. Next, in section III, we derive the continuous and discrete compass-type biped robots by using continuous and discrete mechanics, respectively. In Section IV, we then formulate a gait generation problem for the discrete compass-type biped robot and propose a solving method of it by the sequential quadratic programming to calculate a discrete control input. In addition, we also introduce a transformation method from a discrete control input into a continuous zero-order hold input based on discrete Lagrange-d'Alembert principle. Finally, we show some numerical simulations on gait generation on a periodically unlevel ground for the continuous compass-type biped robot in order to confirm the effectiveness of our method in Section V.

II. Discrete mechanics

In this section, some basic concepts in discrete mechanics are summarized. See [20], [21], [22], [23] for more details on discrete mechanics.

Let Q be an n-dimensional configuration manifold and $q \in \mathbf{R}^n$ be a generalized coordinate of Q. We also refer to T_qQ as the tangent space of Q at a point $q \in Q$ and $\dot{q} \in T_qQ$ denotes a generalized velocity. Moreover, we consider a time-invariant Lagrangian as $L^c(q, \dot{q}) : TQ \to \mathbf{R}$. We first explain about the discretization method. The time variable $t \in \mathbf{R}$ is discretized as $t = kh \ (k = 0, 1, 2, \cdots)$ by using a sampling interval $h > 0$. We denote q_k as a point of Q at the time step k, that is, a curve on Q in the continuous setting is represented as a sequence of points $q^d := \{q_k\}_{k=1}^N$ in the discrete setting. The transformation method of discrete mechanics is carried out by the replacement:

$$q \approx (1-\alpha)q_k + \alpha q_{k+1}, \ \ \dot{q} \approx \frac{q_{k+1} - q_k}{h}, \tag{1}$$

where q is expressed as a internally dividing point of q_k and q_{k+1} with an internal division ratio $\alpha \ (0 < \alpha < 1)$ We then define *a discrete Lagrangian*:

$$L_\alpha^d(q_k, q_{k+1}) := hL\left((1-\alpha)q_k + \alpha q_{k+1}, \frac{q_{k+1} - q_k}{h}\right), \tag{2}$$

and *a discrete action sum*:

$$S_\alpha^d(q_0, q_1, \cdots, q_N) = \sum_{k=0}^{N-1} L_\alpha^d(q_k, q_{k+1}). \tag{3}$$

We next summarize the discrete equations of motion. Consider a variation of points on Q as $\delta q_k \in T_{q_k}Q \ (k = 0, 1, \cdots, N)$ with the fixed condition $\delta q_0 = \delta q_N = 0$. In analogy with the continuous setting, we define a variation of the discrete action sum (3) as

$$\delta S_\alpha^d(q_0, q_1, \cdots, q_N) = \sum_{k=0}^{N-1} \delta L_\alpha^d(q_k, q_{k+1}) \tag{4}$$

as shown in Fig. 1. The discrete Hamilton's principle states that *only a motion which makes the discrete action sum (3) stationary is realized*. Calculating (4), we have

$$\delta S_\alpha^d = \sum_{k=1}^{N-1} \{D_1 L_\alpha^d(q_k, q_{k+1})\delta q_k + D_2 L_\alpha^d(q_{k-1}, q_k)\}\delta q_k, \tag{5}$$

where D_1 and D_2 denotes the partial differential operators with respect to the first and second arguments, respectively. Consequently, from the discrete Hamilton's principle and (5), we obtain *the discrete Euler-Lagrange equations*:

$$\begin{aligned} D_1 L_\alpha^d(q_k, q_{k+1}) + D_2 L_\alpha^d(q_{k-1}, q_k) = 0, \\ k = 1, \cdots, N-1 \end{aligned} \tag{6}$$

with the initial and terminal equations:

$$\begin{aligned} &D_2 L^c(q_0, \dot{q}_0) + D_1 L_\alpha^d(q_0, q_1) = 0 \\ &- D_2 L^c(q_N, \dot{q}_N) + D_2 L_\alpha^d(q_{N-1}, q_N) = 0. \end{aligned} \tag{7}$$

It turns out that (6) is represented as difference equations which contains three points $q_{k-1}, \ q_k, \ q_{k+1}$, and we need $q_0, \ q_1$ as initial conditions when we simulate (6).

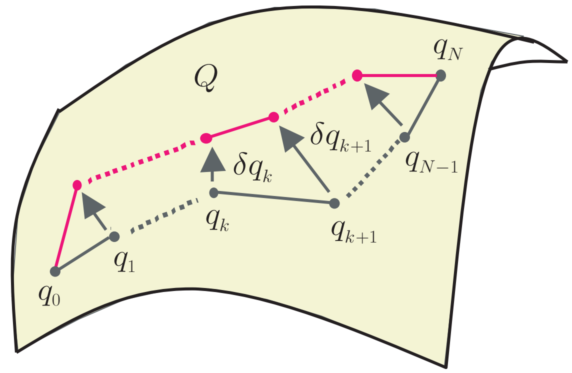

Figure 1 : Discrete Hamilton's principle

Then, we consider a method to add external forces to the discrete Euler-Lagrange equations. By an analogy of continuous mechanics, we denote discrete external forces by $F_\alpha^d : Q \times Q \to T^*(Q \times Q)$, and discretize continuous Lagrange-d'Alembert's principle as

$$\delta \sum_{k=0}^{N-1} L_\alpha^d(q_k, q_{k+1}) + \sum_{k=0}^{N-1} F_\alpha^d(q_k, q_{k+1}) \cdot (\delta q_k, \delta q_{k+1}) = 0, \tag{8}$$

where we define *right/left discrete external forces*: $F_\alpha^{d+}, F_\alpha^{d-} : Q \times Q \to T^*Q$ as

$$\begin{aligned} F_\alpha^{d+}(q_k, q_{k+1})\delta q_k &= F_\alpha^d(q_k, q_{k+1}) \cdot (\delta q_k, 0), \\ F_\alpha^{d-}(q_k, q_{k+1})\delta q_{k+1} &= F_\alpha^d(q_k, q_{k+1}) \cdot (0, \delta q_{k+1}), \end{aligned} \tag{9}$$

respectively. By right/left discrete external forces, a continuous external force $F^c : TQ \to T^*Q$ can be discretized as

$$\begin{aligned} F_\alpha^{d+}(q_k, q_{k+1}) &= (1-\alpha)hF^c\left((1-\alpha)q_k + \alpha q_{k+1}, \frac{q_{k+1} - q_k}{h}\right), \\ F_\alpha^{d-}(q_k, q_{k+1}) &= \alpha hF^c\left((1-\alpha)q_k + \alpha q_{k+1}, \frac{q_{k+1} - q_k}{h}\right). \end{aligned} \tag{10}$$

Therefore, by calculating variations for (8), we obtain *the discrete Euler-Lagrange equations with discrete external forces*:

$$\begin{aligned} D_1 L^d(q_k, q_{k+1}) &+ D_2 L^d(q_{k-1}, q_k) \\ &+ F_\alpha^{d+}(q_k, q_{k+1}) + F^{d-}(q_{k-1}, q_k) = 0, \\ &k = 1, \cdots, N-1, \end{aligned} \tag{11}$$

with the initial and terminal equations:

$$\begin{aligned} &D_2 L^c(q_0, \dot{q}_0) + D_1 L_\alpha^d(q_0, q_1) + F_\alpha^{d+}(q_0, q_1) = 0 \\ &- D_2 L^c(q_N, \dot{q}_N) + D_2 L_\alpha^d(q_{N-1}, q_N) + F_\alpha^{d-}(q_{N-1}, q_N) = 0. \end{aligned} \tag{12}$$

III. Continuous and Discrete Compass-Type Biped Robots

A. *Setting of compass-type biped robot*

In this subsection, we first give a problem setting of the compass-type biped robot. In this paper, we consider a simple compass-type biped robot which consists of two rigid bars (Leg 1 and 2) and a joint without rotational friction (Waist) as shown in Fig. 2. In Fig. 2, Leg 1 is called *the supporting leg* which connects to ground and Leg 2 is called *the swing leg* which is ungrounded. Moreover, for the sake of simplicity, we give the following assumptions; (i) the supporting leg does not slip at the contact point with the ground, (ii) the swing leg hits the ground with completely inelastic collision, (iii) the compass-type biped robot is supported by two legs for just a moment, (iv) the length of the swing leg gets smaller by infinitely small when the swing leg and the supporting leg pass each other.

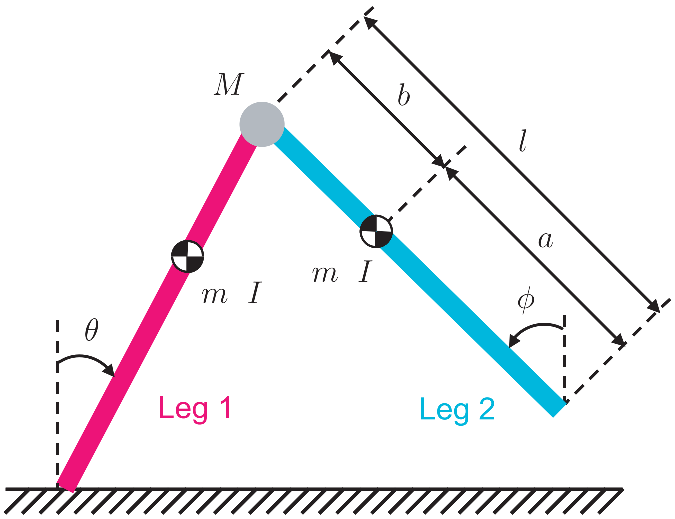

Figure 2 : Compass-type biped robot

Let θ and ϕ be the angles of Leg 1 and 2, respectively. We also use the notations: m: the mass of the legs, M: the mass of the waist, I: the inertia moment of the legs, a: the length between the waist and the center of gravity, b: the length between the center of gravity and the toe of the leg, $l(=a+b)$: the length between the waist and the toe of the leg.

In the walking process of the compass-type biped robot, there exist two modes: *the swing phase* and *the impact phase*. In the swing phase the swing leg is ungrounded, and in the impact phase the toe of the swing leg hit the ground. As shown in Fig. 3, it is noted that the swing phase and the impact phase occur alternately and the swing leg and the supporting leg switch positions with each other with respect to each collision. We denote the order of the swing phase and the impact phase by $i = 1, 2, \cdots, L$ and $i = 1, 2, \cdots, L-1$, respectively. In addition, we assume that Leg 1 is the swing leg and Leg 2 is the supporting leg in odd-numbered swing phases, and Leg 1 is the supporting leg and Leg 2 is the swing leg in even-numbered swing phases.

B. Continuous compass-type biped robot (CCBR)

In this subsection, we derive a model of *continuous compass-type biped robot (CCBR)* via usual continuous mechanics. We denote the angles of Leg 1 and 2 in the i-th swing phase by $\theta^{(i)}$, $\phi^{(i)}$, respectively. In addition, $\dot{\theta}^{(i)}$, $\dot{\phi}^{(i)}$ denote their angular velocities.

First, we consider a model of the CCBR in the i-th swing phase where Leg 1 is the supporting leg and Leg 2 is the swing leg. We assume that the torque at the waist can be controlled, and denote it by $v^{(i)} \in \mathbf{R}$. The Lagrangian of this system L^c is given by (13). Substituting the Lagrangian (13) into the Euler-Lagrange equations and adding the control input to the right-hand sides of them, we have the model of the CCBR in the i-th swing phase as (14), (15). We then derive a model of the CCBR in the i-th impact phase. It is assumed that the swing leg hits the ground with completely inelastic collision, and $\theta^{(i)} = \theta^{(i+1)}$, $\phi^{(i)} = \phi^{(i+1)}$ holds because of an instantaneous impact. Hence, calculating the principle of conservation of angular momentum for the CCBR, we obtain the model of the CCBR in the i-th impact phase as (16), where $a^-, a^+ \in \mathbf{R}^{2\times 2}$ are the coefficient matrices defined by (17) and (18).

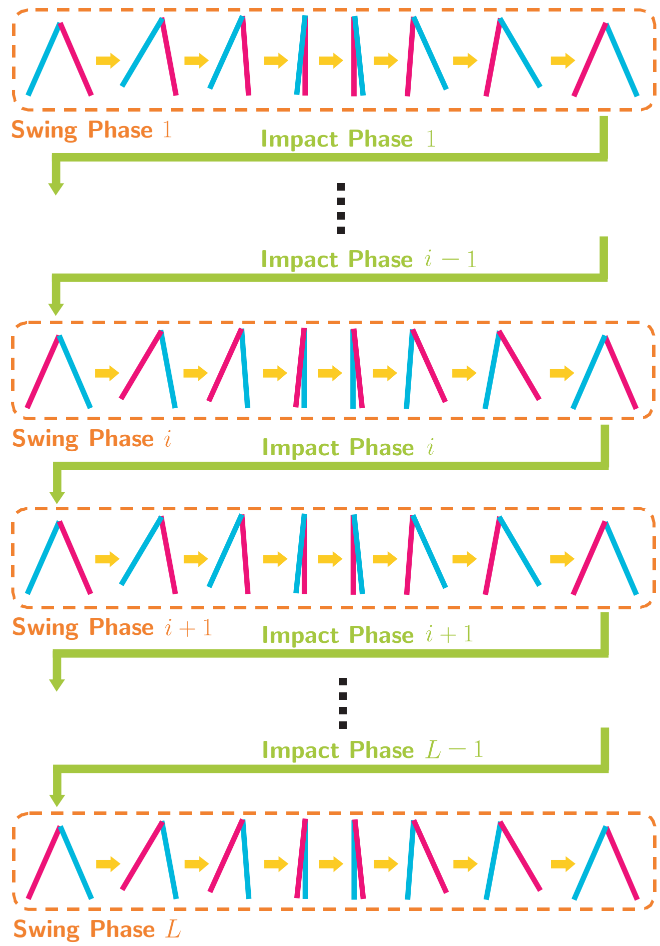

Figure 3 : Gait of compass-type biped robot

C. Discrete compass-type biped robot (DCBR)

Next, we derive a model of *discrete compass-type biped robot (CCBR)* by discrete mechanics in this subsection. We here use the notations; h: the sampling time, $k = 1, 2, \cdots, N$: the time step, $i = 1, \cdots, L$: the order of the swing phases, $\alpha = 1/2$: the internal division ratio in discrete mechanics, $\theta_k^{(i)}$, $\phi_k^{(i)}$: the angles of Leg 1 and 2 at the k-th step in the i-th swing phase.

In this paper, we use only the model of the DCBR in the swing phases, and hence we will derive it. By using the transformation law from a continuous Lagrangian into a discrete Lagrangian (2), we obtain the discrete Lagrangian as (19) from (13). Since the left and right discrete external forces (9) satisfy $F^{d+}(q_k, q_{k+1}) = F^{d-}(q_k, q_{k+1})$ for $\alpha = 1/2$, we

$$L^c(\theta^{(i)},\dot\theta^{(i)},\phi^{(i)},\dot\phi^{(i)}) = \frac{1}{2}(I+ma^2+Ml^2+ml^2)(\dot\theta^{(i)})^2+\frac{1}{2}(I+mb^2)(\dot\phi^{(i)})^2 \\ -mbl\cos(\theta^{(i)}-\phi^{(i)})\dot\theta^{(i)}\dot\phi^{(i)}-(ma+mg+Ml)g\cos\phi^{(i)}+mgb\cos\phi^{(i)} \tag{13}$$

$$\frac{d}{dt}\left(\frac{\partial L^c(\theta^{(i)},\dot\theta^{(i)},\phi^{(i)},\dot\phi^{(i)})}{\partial\dot\theta^{(i)}}\right)-\frac{\partial L^c(\theta^{(i)},\dot\theta^{(i)},\phi^{(i)},\dot\phi^{(i)})}{\partial\theta^{(i)}}=v^{(i)} \tag{14}$$

$$\frac{d}{dt}\left(\frac{\partial L^c(\theta^{(i)},\dot\theta^{(i)},\phi^{(i)},\dot\phi^{(i)})}{\partial\dot\phi^{(i)}}\right)-\frac{\partial L^c(\theta^{(i)},\dot\theta^{(i)},\phi^{(i)},\dot\phi^{(i)})}{\partial\phi^{(i)}}=-v^{(i)} \tag{15}$$

$$a^-(\theta^{(i)},\phi^{(i)})\begin{bmatrix}\dot\theta^{(i)}\\ \dot\phi^{(i)}\end{bmatrix}=a^+(\theta^{(i)},\phi^{(i)})\begin{bmatrix}\dot\theta^{(i+1)}\\ \dot\phi^{(i+1)}\end{bmatrix} \tag{16}$$

$$a^- := \begin{bmatrix}-(2mal+Ml^2)\cos(\theta^{(i)}-\phi^{(i)})+mbl-I & mab-I\\ mab-I & 0\end{bmatrix}, \tag{17}$$

$$a^+ := \begin{bmatrix}-mb^2+mbl\cos(\theta^{(i+1)}-\phi^{(i+1)})-I & -(2ma^2+Ml^2)+mbl\cos(\theta^{(i+1)}-\phi^{(i+1)})-I\\ -mb^2-I & mbl\cos(\theta^{(i+1)}-\phi^{(i+1)})\end{bmatrix}. \tag{18}$$

$$L^d(\theta_k^{(i)},\theta_{k+1}^{(i)},\phi_k^{(i)},\phi_{k+1}^{(i)})=\frac{1}{2}(I+ma^2+Ml^2+ml^2)\left(\frac{\theta_{k+1}^{(i)}-\theta_k^{(i)}}{h}\right)^2+\frac{1}{2}(I+mb^2)\left(\frac{\phi_{k+1}^{(i)}-\phi_k^{(i)}}{h}\right)^2 \\ -mbl\cos\left(\frac{\theta_k^{(i)}+\theta_{k+1}^{(i)}}{2}-\frac{\phi_k^{(i)}+\phi_{k+1}^{(i)}}{2}\right)\frac{\theta_{k+1}^{(i)}-\theta_k^{(i)}}{h}\frac{\phi_{k+1}^{(i)}-\phi_k^{(i)}}{h} \\ -(ma+mg+Ml)g\cos\left(\frac{\phi_k^{(i)}+\phi_{k+1}^{(i)}}{2}\right)+mgb\cos\left(\frac{\phi_k^{(i)}+\phi_{k+1}^{(i)}}{2}\right) \tag{19}$$

$$D_2L^d(\theta_{k-1}^{(i)},\theta_k^{(i)},\phi_{k-1}^{(i)},\phi_k^{(i)})-D_1L^d(\theta_k^{(i)},\theta_{k+1}^{(i)},\phi_k^{(i)},\phi_{k+1}^{(i)})+u_{k-1}^{(i)}+u_k^{(i)}=0\quad(k=2,\cdots,N) \tag{21}$$

$$D_4L^d(\theta_{k-1}^{(i)},\theta_k^{(i)},\phi_{k-1}^{(i)},\phi_k^{(i)})-D_3L^d(\theta_k^{(i)},\theta_{k+1}^{(i)},\phi_k^{(i)},\phi_{k+1}^{(i)})-u_{k-1}^{(i)}-u_k^{(i)}=0\quad(k=2,\cdots,N) \tag{22}$$

$$D_2L^c(\theta_1^{(i)},\dot\theta_1^{(i)},\phi_1^{(i)},\dot\phi_1^{(i)})+D_1L^d(\theta_1^{(i)},\theta_2^{(i)},\phi_1^{(i)},\phi_2^{(i)})+u_1^{(i)}=0\quad(k=2,\cdots,N) \tag{23}$$

$$D_4L^c(\theta_1^{(i)},\dot\theta_1^{(i)},\phi_1^{(i)},\dot\phi_1^{(i)})+D_3L^d(\theta_1^{(i)},\theta_2^{(i)},\phi_1^{(i)},\phi_2^{(i)})-u_1^{(i)}=0\quad(k=2,\cdots,N) \tag{24}$$

$$-D_2L^c(\theta_N^{(i)},\dot\theta_N^{(i)},\phi_N^{(i)},\dot\phi_N^{(i)})+D_1L^d(\theta_{N-1}^{(i)},\theta_N^{(i)},\phi_{N-1}^{(i)},\phi_N^{(i)})+u_{N-1}^{(i)}=0 \tag{25}$$

$$-D_4L^c(\theta_N^{(i)},\dot\theta_N^{(i)},\phi_N^{(i)},\dot\phi_N^{(i)})+D_3L^d(\theta_{N-1}^{(i)},\theta_N^{(i)},\phi_{N-1}^{(i)},\phi_N^{(i)})-u_{N-1}^{(i)}=0 \tag{26}$$

$$a^-(\theta_N^{(i)},\phi_N^{(i)})\begin{bmatrix}\dot\theta_N^{(i)}\\ \dot\phi_N^{(i)}\end{bmatrix}=a^+(\theta_N^{(i)},\phi_N^{(i)})\begin{bmatrix}\dot\theta_1^{(i+1)}\\ \dot\phi_1^{(i+1)}\end{bmatrix}, \tag{27}$$

set a discrete control input that consists of only the left discrete external force F^{d-} as

$$u_k^{(i)} := F^{d-}(q_k,q_{k+1}),\ \ k=1,\cdots,N-1. \tag{20}$$

Then, substituting (19) into the discrete Euler-Lagrange equations (11), (12) and using the discrete control input (20), we have the model of the DCBR in the i-th swing phase as (21)–(26).

For the impact phases, we use the model of the CCBR (16), and we rewrite it with the terminal variables of the i-the swing phase $\theta_N^{(i)}$ $\phi_N^{(i)}$, $\dot\theta_N^{(i)}$ $\dot\phi_N^{(i)}$ and the initial variables of the $(i+1)$-the swing phase $\theta_1^{(i+1)}$ $\phi_1^{(i+1)}$, $\dot\theta_1^{(i+1)}$ $\dot\phi_1^{(i+1)}$ as (27). This representation (27) will be utilized in the next section.

IV. Gait generation method on periodically unlevel grounds

A. Setting of periodically unlevel grounds

First, this subsection formulates the problem setting of grounds on which the compass-type biped robot walks. As shown in Fig. 4, set the x and z axes to the horizontal and vertical directions, respectively, and P_0 denotes the origin of the xz-plane. We also set L points: $P_1, P_2, \cdots, P_L$ in the xz-plane, and represent P_i as $P_i = (r_i, \rho_i)$ by using the polar coordinate with reference to P_{i-1} as illustrated in Fig. 5. Note that $r_i > 0$, $-\pi/2 < \rho_i < \pi/2$ are assumed. The sequence of points $P_1, P_2, \cdots, P_L$ are reference grounding points for the compass-type biped robot as shown in Fig. 6.

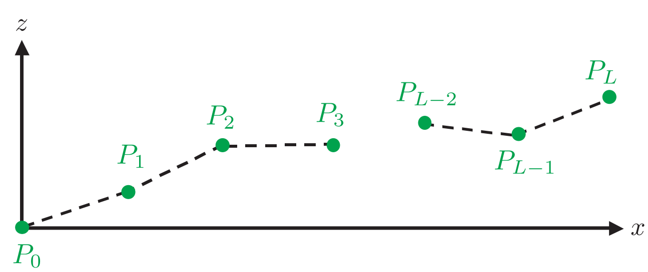

Figure 4 : Reference grounding points in xz-plane

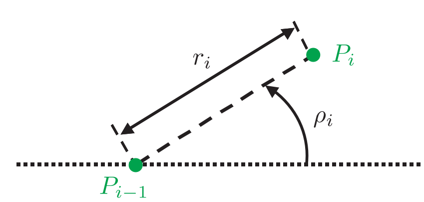

Figure 5 : r_i and ρ_i

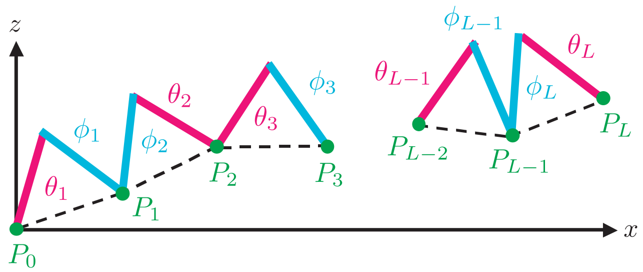

Figure 6 : Desired gait of compass-type biped robot

This problem setting can treat various walking surfaces, for example, flats [30]: $\rho_i = 0$ $(i = 1, \cdots, L)$, downward slopes [32]: $\rho_i = \rho^- < 0$ $(i = 1, \cdots, L)$, and upward slopes in [32]: $\rho_i = \rho^+ > 0$ $(i = 1, \cdots, L)$. In this paper, we consider gait generation on periodically unlevel grounds as depicted in Fig. 7 with the parameter:

$$\rho_i = \begin{cases} \rho, & i = 1, 3, 5, \cdots, L-1, \\ -\rho, & i = 2, 4, 6, \cdots, L, \end{cases} \tag{28}$$

where $\rho > 0$ and L is an odd number. Since a periodically unlevel ground contains both downward and upward slopes, this type of gait generation problems is expected to be more difficult to solve in comparison with the downward and upward slopes cases [32]. Based on the setting above, we consider the following problem on the gait generation for the CCBR.

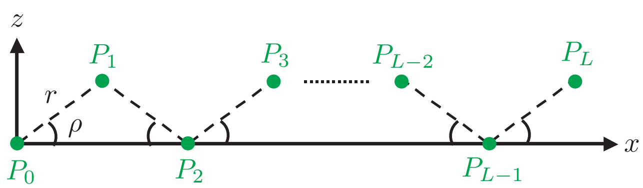

Figure 7 : Setting of Periodically Unlevel Grounds

Problem 1: For the continuous compass-type biped robot (CCBR) (14)–(16), find a control input $v^{(i)}$ $(i = 1, \cdots, L)$ such that the swing leg of the CCBR lands at the reference grounding points P_i $(i = 1, \cdots, L)$ on a periodically unlevel ground of (28) with a stable and natural gait. □

In order to solve Problem 1 above, we shall consider a method based on discrete mechanics. The method consists of two steps: (i) calculation of a discrete control input by solving a finite dimensional constrained nonlinear optimization problem (Subsection IV-B), (ii) transformation a discrete control input into a zero-order hold input by discrete Lagrange-d'Alembert principle (Subsection IV-C).

B. Gait generation problem for the DCBR

As the first step, we consider a problem on generation of a discrete gait for the DCBR in stead of the CCBR. The discrete gait generation problem for the DCBR is stated as follows.

Problem 2: For the discrete compass-type biped robot (DCBR) (21)–(26), find a sequence of the control input $u_k^{(i)}$ $(i = 1, \cdots, L,\ k = 1, \cdots, N-1)$ such that the swing leg of the DCBR lands at the reference grounding points P_i $(i = 1, \cdots, L)$ with a stable and natural discrete gait. □

Our main purpose is that we obtain the mathematical formulation of Problem 2 as an optimal control problem. In order to do this, we focus attention on a periodical motions of the DCBR. It must be noted that the DCBR walks on upward and downward slopes alternately, and hence we consider one upward slope and one downward slope as a set (see Fig. 8). If the initial angular velocities of the swing leg at the i-th and $(i+2)$-th swing phases are pretty much the same, a stable gait of the DCBR can be generated as shown in Fig. 8. So, we introduce a cost function of a square of difference between initial angular velocities in the i-th and $(i+2)$-th swing phases:

$$J = (\dot{\phi}_1^{(i+2)} - \dot{\phi}_1^{(i)})^2 + (\dot{\theta}_1^{(i+2)} - \dot{\theta}_1^{(i)})^2. \tag{29}$$

However, the cost function (29) contains the angular velocities in the $(i+2)$-th swing phase. To avoid this, we eliminate $\dot{\phi}_1^{(i+2)}$, $(\dot{\theta}_1^{(i+2)}$ by using the $(i+1)$-th impact phase model [1]:

$$\begin{aligned} a_-^{(i+1)}(\phi_N^{(i+1)}, \theta_N^{(i+1)}) \begin{bmatrix} \dot{\phi}_N^{(i+1)} \\ \dot{\theta}_N^{(i+1)} \end{bmatrix} \\ = a_+^{(i+1)}(\phi_N^{(i+1)}, \theta_N^{(i+1)}) \begin{bmatrix} \dot{\phi}_1^{(i+2)} \\ \dot{\theta}_1^{(i+2)} \end{bmatrix}. \end{aligned} \tag{30}$$

Solving (30) for $\dot{\phi}_1^{(i+2)}$ and $\dot{\theta}_1^{(i+2)}$, and substituting this into the cost function (29), we have

$$\begin{aligned} J = & (a_{11}^{(i+1)}\dot{\phi}_N^{(i+1)} + a_{12}^{(i+1)}\dot{\theta}_N^{(i+1)} - \dot{\phi}_1^{(i)})^2 \\ & + (a_{21}^{(i+1)}\dot{\phi}_N^{(i+1)} + a_{22}^{(i+1)}\dot{\theta}_N^{(i+1)} - \dot{\theta}_1^{(i)})^2 \end{aligned} \tag{31}$$

where

$$(a_+^{(i+1)})^{-1} a_-^{(i+1)} =: \begin{bmatrix} a_{11}^{(i+1)} & a_{12}^{(i+1)} \\ a_{21}^{(i+1)} & a_{22}^{(i+1)} \end{bmatrix}.$$

We can see that the new cost function (31) does not contain $\dot{\phi}_1^{(i+2)}$ $\dot{\theta}_1^{(i+2)}$ and is represented by only variables in the i-th and $(i+1)$-th swing phases. Consequently, Problem 2 can be formulated as (32)–(48). In the optimization control problem (32)–(48), (32) is the cost function to be minimized, (33)–(38) are the i-th swing phase model, (39)–(44) are the $(i+1)$-th swing phase model, and (46) is the i-th impact phase model. Moreover, (46) and (47) indicates constraints that prevent a reverse behavior of the swing leg and realize a natural gait. (48) are given data on initial and desired angles of Leg 1 and 2, which can be obtained from data of the reference grounding points P_i $(i = 1, \cdots, N)$,

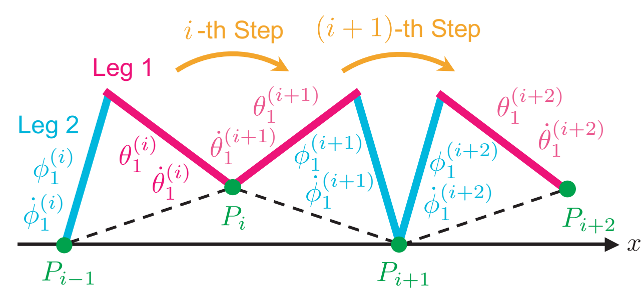

Figure 8 : A gait on a periodically unlevel ground.

[1] Since Leg 1 is the swing leg and Leg 2 is the supporting one in the $(i+1)$-th swing phase, the $(i+1)$-th impact model can be obtained by exchanging θ for ϕ in (27).

$$
\begin{aligned}
\text{min} \quad & J = (a_{11}^{(i+1)}\dot{\phi}_N^{(i+1)} + a_{12}^{(i+1)}\dot{\theta}_N^{(i+1)} - \dot{\phi}_1^{(i)})^2 + (a_{21}^{(i+1)}\dot{\phi}_N^{(i+1)} + a_{22}^{(i+1)}\dot{\theta}_N^{(i+1)} - \dot{\theta}_1^{(i)})^2 && (32)\\
\text{s.t.} \quad & D_2L^d(\theta_{k-1}^{(i)},\theta_k^{(i)},\phi_{k-1}^{(i)},\phi_k^{(i)}) - D_1L^d(\theta_k^{(i)},\theta_{k+1}^{(i)},\phi_k^{(i)},\phi_{k+1}^{(i)}) + u_{k-1}^{(i)} + u_k^{(i)} = 0 \ (k=2,\cdots,N) && (33)\\
& D_4L^d(\theta_{k-1}^{(i)},\theta_k^{(i)},\phi_{k-1}^{(i)},\phi_k^{(i)}) - D_3L^d(\theta_k^{(i)},\theta_{k+1}^{(i)},\phi_k^{(i)},\phi_{k+1}^{(i)}) - u_{k-1}^{(i)} - u_k^{(i)} = 0 \ (k=2,\cdots,N) && (34)\\
& D_2L^d(\theta_1^{(i)},\dot{\theta}_1^{(i)},\phi_1^{(i)},\dot{\phi}_1^{(i)}) + D_1L^d(\theta_1^{(i)},\theta_2^{(i)},\phi_1^{(i)},\phi_2^{(i)}) + u_1^{(i)} = 0 \ (k=2,\cdots,N) && (35)\\
& D_4L^d(\theta_1^{(i)},\dot{\theta}_1^{(i)},\phi_1^{(i)},\dot{\phi}_1^{(i)}) + D_3L^d(\theta_1^{(i)},\theta_2^{(i)},\phi_1^{(i)},\phi_2^{(i)}) - u_1^{(i)} = 0 \ (k=2,\cdots,N) && (36)\\
& -D_2L^c(\theta_N^{(i)},\dot{\theta}_N^{(i)},\phi_N^{(i)},\dot{\phi}_N^{(i)}) + D_1L^d(\theta_{N-1}^{(i)},\theta_N^{(i)},\phi_{N-1}^{(i)},\phi_N^{(i)}) + u_{N-1}^{(i)} = 0 && (37)\\
& -D_4L^c(\theta_N^{(i)},\dot{\theta}_N^{(i)},\phi_N^{(i)},\dot{\phi}_N^{(i)}) + D_3L^d(\theta_{N-1}^{(i)},\theta_N^{(i)},\phi_{N-1}^{(i)},\phi_N^{(i)}) - u_{N-1}^{(i)} = 0 && (38)\\
& D_2L^d(\theta_{k-1}^{(i+1)},\theta_k^{(i+1)},\phi_{k-1}^{(i+1)},\phi_k^{(i+1)}) - D_1L^d(\theta_k^{(i+1)},\theta_{k+1}^{(i+1)},\phi_k^{(i+1)},\phi_{k+1}^{(i+1)}) + u_{k-1}^{(i+1)} + u_k^{(i+1)} = 0 \ (k=2,\cdots,N) && (39)\\
& D_4L^d(\theta_{k-1}^{(i+1)},\theta_k^{(i+1)},\phi_{k-1}^{(i+1)},\phi_k^{(i+1)}) - D_3L^d(\theta_k^{(i+1)},\theta_{k+1}^{(i+1)},\phi_k^{(i+1)},\phi_{k+1}^{(i+1)}) - u_{k-1}^{(i+1)} - u_k^{(i+1)} = 0 \ (k=2,\cdots,N) && (40)\\
& D_2L^d(\theta_1^{(i+1)},\dot{\theta}_1^{(i+1)},\phi_1^{(i+1)},\dot{\phi}_1^{(i+1)}) + D_1L^d(\theta_1^{(i+1)},\theta_2^{(i+1)},\phi_1^{(i+1)},\phi_2^{(i+1)}) + u_1^{(i+1)} = 0 \ (k=2,\cdots,N) && (41)\\
& D_4L^d(\theta_1^{(i+1)},\dot{\theta}_1^{(i+1)},\phi_1^{(i+1)},\dot{\phi}_1^{(i+1)}) + D_3L^d(\theta_1^{(i+1)},\theta_2^{(i+1)},\phi_1^{(i+1)},\phi_2^{(i+1)}) - u_1^{(i+1)} = 0 \ (k=2,\cdots,N) && (42)\\
& -D_2L^c(\theta_N^{(i+1)},\dot{\theta}_N^{(i+1)},\phi_N^{(i+1)},\dot{\phi}_N^{(i+1)}) + D_1L^d(\theta_{N-1}^{(i+1)},\theta_N^{(i+1)},\phi_{N-1}^{(i+1)},\phi_N^{(i+1)}) + u_{N-1}^{(i+1)} = 0 && (43)\\
& -D_4L^c(\theta_N^{(i+1)},\dot{\theta}_N^{(i+1)},\phi_N^{(i+1)},\dot{\phi}_N^{(i+1)}) + D_3L^d(\theta_{N-1}^{(i+1)},\theta_N^{(i+1)},\phi_{N-1}^{(i+1)},\phi_N^{(i+1)}) - u_{N-1}^{(i+1)} = 0 && (44)\\
& a_-^{(i)}(\theta_N^{(i)},\phi_N^{(i)}) \begin{bmatrix} \dot{\theta}_N^{(i)} \\ \dot{\phi}_N^{(i)} \end{bmatrix} = a_+^{(i)}(\theta_N^{(i)},\phi_N^{(i)}) \begin{bmatrix} \dot{\theta}_1^{(i+1)} \\ \dot{\phi}_1^{(i+1)} \end{bmatrix} && (45)\\
& \phi_1^{(i)} < \phi_2^{(i)} < \cdots < \phi_{N-1}^{(i)} < \phi_N^{(i)} && (46)\\
& \theta_1^{(i+1)} < \theta_2^{(i+1)} < \cdots < \theta_{N-1}^{(i+1)} < \theta_N^{(i+1)} && (47)\\
\text{given} \quad & \theta_1^{(i)},\phi_1^{(i)},\theta_N^{(i)},\phi_N^{(i)},\theta_1^{(i+1)},\phi_1^{(i+1)},\theta_N^{(i+1)},\phi_N^{(i+1)} && (48)
\end{aligned}
$$

It turns out that the optimization control problem (32)–(48) is represented as a finite dimensional constrained nonlinear optimization problem with respect to the $(6N+6)$ variables: $\theta_1^{(i)},\cdots,\theta_N^{(i)}$, $\theta_1^{(i+1)},\cdots,\theta_N^{(i+1)}$, $\phi_1^{(i)},\cdots,\phi_N^{(i)}$, $\phi_1^{(i+1)},\cdots,\phi_N^{(i+1)}$, $u_1^i,\cdots,u_{N-1}^i$, $u_1^{i+1},\cdots,u_{N-1}^{i+1}$, $\dot{\theta}_1^{(i)}$, $\dot{\phi}_1^{(i)}$, $\dot{\theta}_N^{(i)}$, $\dot{\phi}_N^{(i)}$, $\dot{\theta}_1^{(i+1)}$, $\dot{\phi}_1^{(i+1)}$, $\dot{\theta}_N^{(i+1)}$, $\dot{\phi}_N^{(i+1)}$. Therefore, we can solve it by *the sequential quadratic programming* [23], [33], and obtain a sequence of discrete control input $u_1^{(i)},\cdots,u_{N-1}^{(i)}$, $u_1^{(i+1)},\cdots,u_{N-1}^{(i+1)}$.

C. Transformation to continuous-time zero-order hold input

The previous subsection presents a synthesis method of a discrete control to generate a discrete gait of the DCBR by solving a finite dimensional constrained nonlinear optimization problem. However, since the control input is discrete-time, it cannot be utilized for the CCBR. Therefore, we here consider transformation of a discrete control input into a continuous one.

There exist infinite methods to generate a continuous control input from a given discrete one, and a continuous control input generated from a given discrete input has to be consistent with laws of physics. Hence, in this paper, we deal with a zero-order hold input in the form:

$$v^{(i)}(t) = v_k^{(i)}, \ (i-1)kh \le t < (i-1)(k+1)h, \qquad (49)$$

which is one of the simplest continuous inputs. We need to derive a relationship between a discrete input $u_k^{(i)}$ $(k = 1,2,\cdots,N-1)$ and a zero-order hold input (49). By using discrete Lagrange-d'Alembert's principle which is explained in Section II, we can have the following theorem.

Theorem 1: A zero-order hold input (49) that satisfies discrete Lagrange-d'Alembert's principle is given by

$$v_k^{(i)} = \frac{2}{h}u_k^{(i)}. \qquad (50)$$

(Proof) During the time interval $(i-1)kh \le t < (i-1)(k+1)h$, substituting (20) and (49) into the definition of the left discrete external force in (9):

$$F^{d-}(q_k,q_{k+1}) = \frac{h}{2}F^c\left((1-\alpha)q_k + \alpha q_{k+1}, \frac{q_{k+1}-q_k}{h}\right),$$

we obtain

$$u_k^{(i)} = \frac{h}{2}v_k^{(i)}.$$

Hence, we have (50). □

By using (50) in Theorem 1, we can easily calculate a zero-order hold input from $u_k^{(i)}$, $i = 1,\cdots,N-1$ which are obtained by solving a finite dimensional constrained nonlinear optimization problem (32)–(48). In addition, it must be noted that since we use discrete Lagrange-d'Alembert's principle to prove Theorem 1, a zero-order hold input with a gain (50) is consistent with laws of physics.

V. NUMERICAL SIMULATIONS

A. Problem formulation

In this section, some numerical simulations on a gait generation on a periodically unlevel ground for the CCBR based on our new method proposed in the previous section, and confirm the effectiveness of it. First, this subsection gives the problem setting. we set parameters as follows; the physical parameters of the CCBR: $m = 2.0$ [kg], $M = 10.0$ [kg], $I =$

$0.167\,[\mathrm{kgm}^2]$, $a = 0.5\,[\mathrm{m}]$, $b = 0.5\,[\mathrm{m}]$, $l = 1.0\,[\mathrm{m}]$, the parameters of discrete mechanics: $\alpha = 1/2$, $h = 0.005\,[\mathrm{s}]$, $N = 80$. We consider two types of periodically unlevel grounds. The one is set as $r = 1.0\,[\mathrm{m}]$, $\rho = 5\,[\mathrm{deg}]$, $L = 8$ (Simulation I), and the other is set as $r = 1.0\,[\mathrm{m}]$, $\rho = 10\,[\mathrm{deg}]$, $L = 8$ (Simulation II). Intial conditions are $\theta_1^{(1)} = -0.5321\,[\mathrm{rad}]$, $\phi_1^{(1)} = 2.0273\,[\mathrm{rad}]$, $\dot\theta_1^{(1)} = 0.1830\,[\mathrm{rad}]$, $\dot\phi_1^{(1)} = 2.1820\,[\mathrm{rad}]$.

B. Simulation results

Next, numerical simulations are shown in order to check the availability of our new approach. Figs. 9–11 show the results of Simulation I. Fig. 9 illustrates the time series of Leg 1 and 2 (θ and ϕ). Fig. 10 shows the plot of solution trajectory in the phase space of $\theta - \phi$. In Fig. 11, a snapshot of the continuous gait is depicted. From these results, we can confirm that a stable gait on periodically unlevel grounds for the CCBR can be generated by the proposed approach.

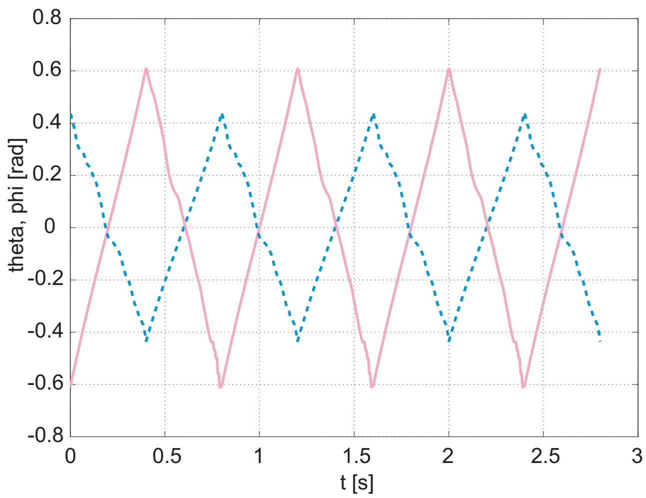

Figure 9 : Time series of θ and ϕ (Simulation I; red line: θ, blue line: ϕ)

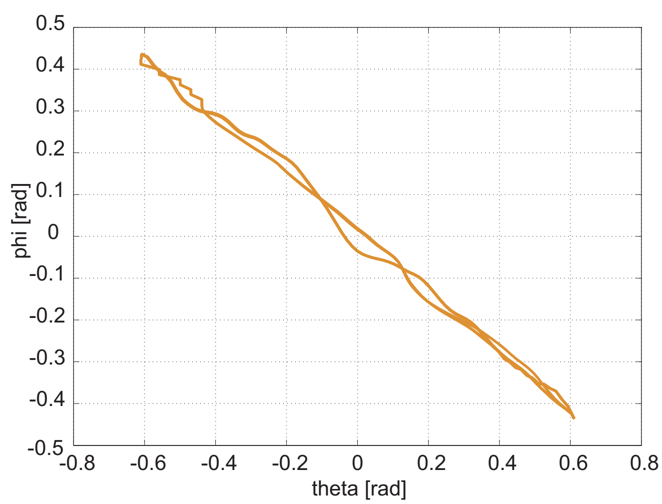

Figure 10 : Solution trajectory on $\theta\,\phi$-space (Simulation I)

Figs. 12–14 illustrate the results of Simulation II. Fig. 12 depicts the time series of Leg 1 and 2 (θ and ϕ). Fig. 13 shows the plot of solution trajectory in the phase space of $\theta - \phi$. In Fig. 14, a snapshot of the continuous gait is illustrated. From these results, we can also see that the proposed method can generate a stable gait for the CCBR.

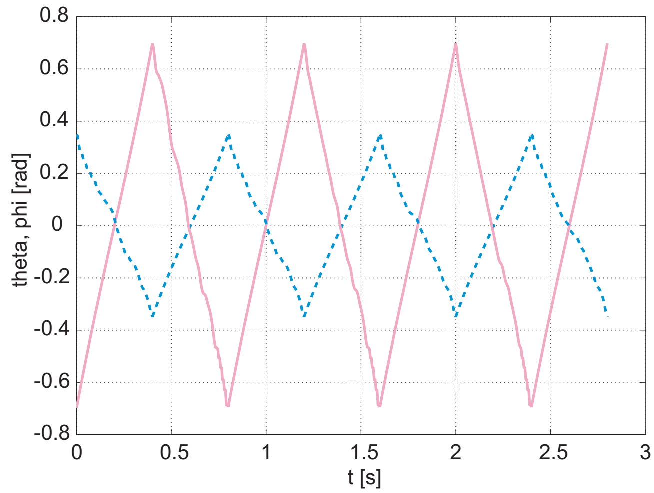

Figure 12 : Time series of θ and ϕ (Simulation II; red line: θ, blue line: ϕ)

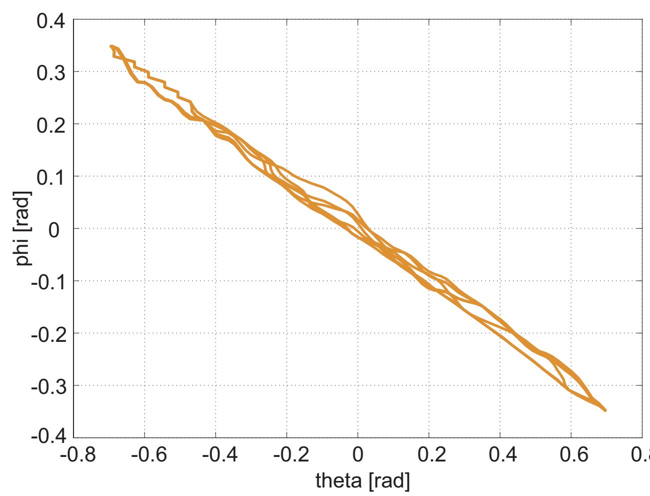

Figure 13 : Solution trajectory on $\theta\,\phi$-space (Simulation II)

VI. Conclusions

This paper has dealt with a gait generation problem for the compass-type biped robot on periodically unlevel grounds. We have formulated a discrete gait generation problem for the DCBR as a finite dimensional constrained nonlinear optimization problem. A transformation method from a discrete control input into a zero-order hold input has been introduced from the viewpoint of discrete Lagrange-d'Alembert principle. By numerical simulations, we have verified generation of a stable gait and the effectiveness of our new approach.

In association with this work, we will tackle the following problems: stable gait generation of the CCBR irregular grounds, experimental evaluation of the proposed control method, and applications of discrete mechanics to more human-like robots.

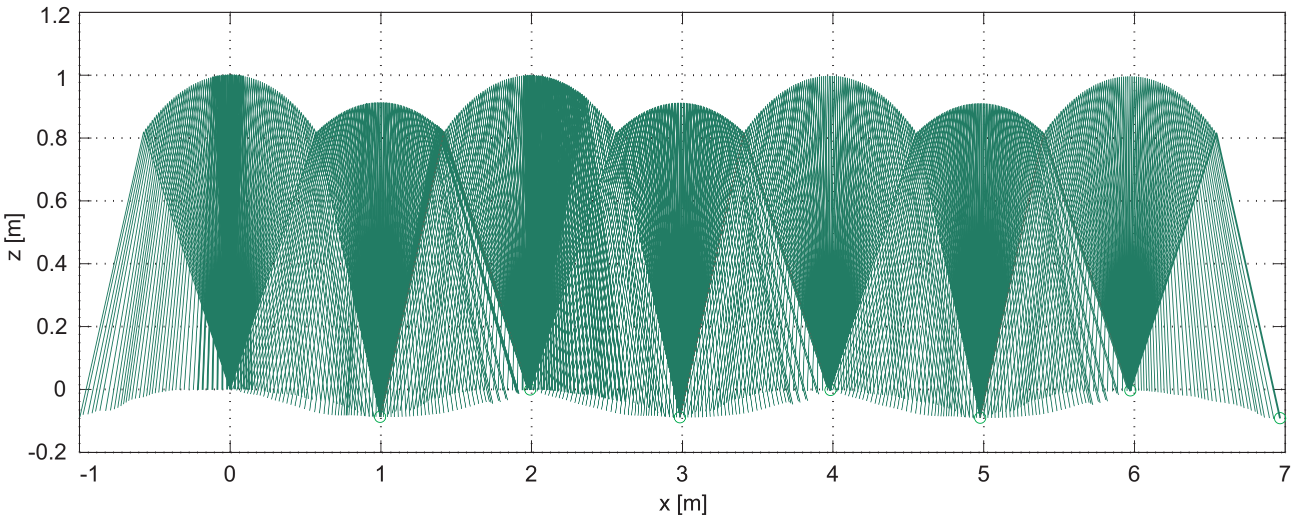

Fig. 11 : Snapshot of Gait (Simulation I)

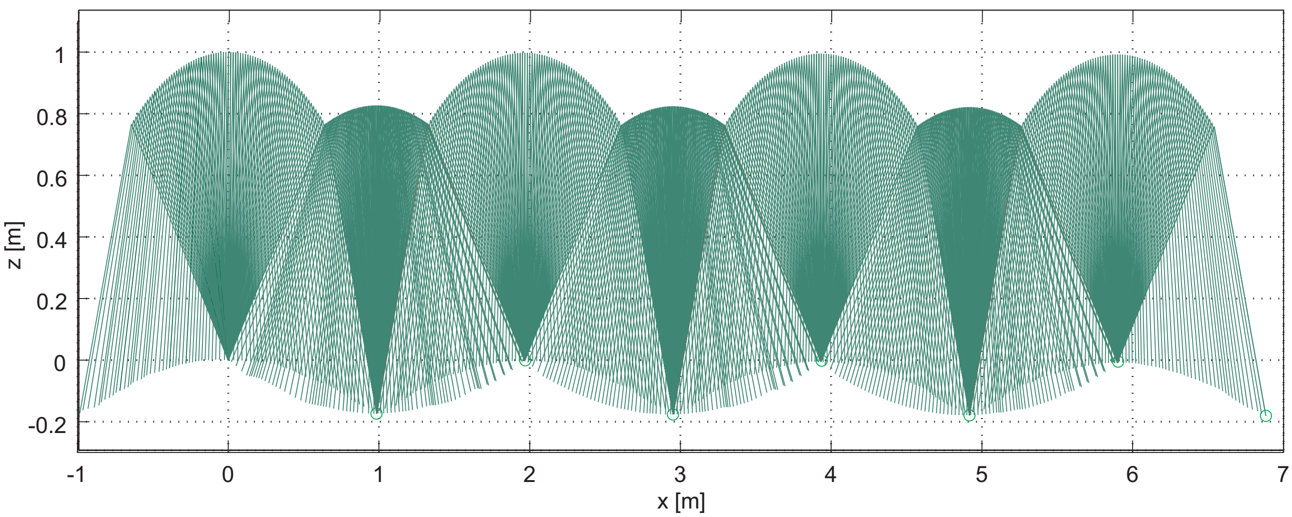

Fig. 14 : Snapshot of Gait (Simulation II)

REFERENCES

[1] T. McGeer, "Passive Dynamic Walking," *The International Journal of Robotics Research*, Vol. 9, No. 2, pp. 62-82, 1990.

[2] A. Goswami, B. Thuilot, and B. Espiau, "A Study of the Passive Gait of a Compass-like Biped Robot: Symmetry and Chaos," *International Journal of Robotics Research*, Vol. 17, No. 12, pp. 1282-1301, 1998.

[3] S. H. Collins, A. Ruina, R. Tedrake, and M. Wisse, "Efficient Bipedal Robots based on Passive-Dynamic Walkers," *Science*, Vol. 307, pp. 1082–1085, 2005.

[4] H. Gritli, N. Khraief, and S. Belghith, "Period-Three Route to Chaos induced by a Cyclic-Fold Bifurcation in Passive Dynamic Walking of a Compass-Gait Biped Robot," *Communications in Nonlinear Science and Numerical Simulation*, Vol. 17, pp. 4356–4372, 2012.

[5] A. Goswami, B. Espiau and A. Keramane, "Limit Cycles in a Passive Compass Gait Biped and Passivity Mimicking Control Laws," *Autonomous Robots*, Vol. 4, pp. 273-286, 1997.

[6] M. Garcia, A. Chatterjee, A. Ruina, and M. J. Coleman, "The simplest walking model: stability, complexity, and scaling," *ASME Journal of of Biomech. Eng.*, Vol. 120, No. 2, pp. 281-288, 1998.

[7] J. W. Grizzle, G. Abba, and F. Plestan, "Asymptotically Stable Walking for Biped Robots: Analysis via Systems with Impulse Effects," *IEEE Trans. on Automatic Control*, Vol. 46, No. 1, pp. 51-64, 2001.

[8] E. R. Westervelt, J. W. Grizzle, and C. Canudas de Wit, "Switching and PI Control of Walking Motions of Planar Biped Walkers," *IEEE Transactions on Automatic Control*, Vol. 48, No. 2, pp. 308-312, 2003.

[9] J. Hass, J. M. Herrmann, and T. Geisel, "Optimal Mass Distribution for Passivity-based Bipedal Robots," *International Journal of Robotics Research*, Vol. 25, No. 11, pp. 1087–1098, 2006.

[10] A. T. Safa, M. G. Saadat, and M. Naraghi "Passive Dynamic of the Simplest Walking Model: Replacing Ramps with Stairs," *Mechanism and Machine Theory*, Vol. 42, pp. 1314-1325, 2007.

[11] D. G. E. Hobbelen and M. Wisse, "Swing-leg Retraction for Limit Cycle Walkers Improves Disturbance Rejection," *IEEE Transactions on Robotics*, Vol. 24, No. 2, pp. 377–389, 2008.

[12] Q. Huang, K. Yokoi, S. Kajita, K. Kaneko, H. Arai, N. Koyachi, and K. Tanie, "Planning Walking Patterns for a Biped Robot," *IEEE Transactions on Robotics and Automation*, Vol. 17, No. 3, pp. 280-289, 2001.

[13] P. Sardain and G. Bessonnet, "Forces Acting on a Biped Robot. Center

of Pressure – Zero Moment Point," *IEEE Transactions on Systems, Man, and Cybernetics, Part A: Systems and Humans*, Vol. 34, No. 5, pp. 630-637, 2004.

[14] K. Erbatur and O. Kurt, "Natural ZMP Trajectories for Biped Robot Reference Generation," *IEEE Transactions on Industrial Electronics*, Vol. 56, No. 3, pp. 835-845, 2009.

[15] C. M. A. Pinto, A. P. Santos, "Modelling Gait Transition in Two-Legged Animals," *Communications in Nonlinear Science and Numerical Simulation*, Vol. 16, pp. 4625–4631, 2011.

[16] G. Taga, Y. Yamaguchi, and H. Shimizu, "Self-organized Control of Bipedal Locomotion by Neural Oscillators in Unpredictable Environment," *Biological Cybernetics*, Vol. 65, pp. 147–159, 1991.

[17] W. T. Miller, "Real-time Neural Network Control of a Biped Walking Robot," *IEEE Control Systems Magazine*, Vol. 14, No. 1, pp. 41–48, 1994.

[18] J. Nakanishi, J. Morimoto, G. Endo, G. Cheng, S. Schaal, and M. Kawato, "Learning from Demonstration and Adaptation of Biped Locomotion," *Robotics and Autonomous Systems*, Vol. 47, pp. 79–91, 2004.

[19] J. Morimoto and C. G. Atkeson, "Learning Biped Locomotion: Application of Poincare-map-based reinforcement learning," *IEEE Robotics & Automation Magazine*, Vol. 14, No. 2, pp. 41–51, 2007.

[20] J. E. Marsden, G. W. Patrick, and S. Shkoller, Multisymplectic Geometry, Variational Integrators and Nonlinear PDEs, *Communications in Mathematical Physics*, Vol. 199, pp. 351-395, 1998.

[21] C. Kane, J. E. Marsden, M. Ortiz, and M. West, "Variational Integrators and the Newmark Algorithm for Conservative and Dissipative Mechanical Systems," *International Journal for Numerical Methods in Engineering*, Vol. 49, pp. 1295–1325, 2000.

[22] J. E. Marsden and M. West, "Discrete Mechanics and Variational Integrators," *Acta Numerica*, Vol. 10, pp. 3571-5145, 2001.

[23] O. Junge, J. E. Marsden, and S. Ober-Blöbaum, "Discrete Mechanics and Optimal Control," *Proc. of 16th IFAC World Congress*, Praha, Czech, Paper No. We-M14-TO/3, 2005

[24] A. M. Bloch, M. Leok, J. E. Marsden and D. V. Zenkov, "Controlled Lagrangians and Stabilization of the Discrete Cart-Pendulum System," *Proc. of 44th IEEE CDC-ECC*, Seville, Spain, pp. 6579-6584, 2005.

[25] A. M. Bloch, M. Leok, J. E. Marsden and D. V. Zenkov, "Controlled Lagrangians and Potential Shaping for Stabilization of the Discrete Mechanical Systems," *Proc. of 45th IEEE CDC*, San Diego, U.S.A., pp. 3333-3338, 2006.

[26] T. Kai and Y. Yamamoto, "On Analysis and Control of the Cart-Pendulum System Modeled by Discrete Mechanics," *Proc. of NOLTA 2007*, Vancouver, Canada, pp. 485–488, 2007.

[27] T. Kai and K. Bito, "Solvability Analysis and Stabilization of the Cart-Pendulum Modeled by Discrete Mechanics with Friction," *Proc. of NOLTA 2008*, Budapest, Hungary, pp. 305-308, 2008.

[28] T. Kai and T. Shintani, "Stabilization Control and Experiments of the Cart-Pendulum based on Discrete Mechanics," *Proc. of NOLTA 2009*, Sapporo, Japan, pp. 82-85, 2009.

[29] T. Kai and T. Shintani, "Discrete Gait Generation for the Compass-Type Biped Robot Modeled by Discrete Mechanics," *in Proc. of NOLTA 2010*, Krakow, Poland, pp. 434–437, 2009.

[30] T. Kai and T. Shintani, "Discrete Mechanics Approach to Gait Generation for the Compass-type Biped Robot," *in Proc. of IFAC World Congress 2011*, Milano, Italy, pp. 434–437, 2011.

[31] T. Kai and T. Shintani, "A Discrete Mechanics Approach to Gait Generation for the Compass-Type Biped Robot," *Nonlinear Theory and Its Applications, IEICE*, Vol. 2, No. 4, pp. 533-547, 2011.

[32] T. Kai and T. Shintani, "A Gait Generation Method for the Compass-type Biped Robot on Slopes via Discrete Mechanics," *in Proc. of IEEE CDC 2011*, Orlando, USA, pp. 675–681, 2011.

[33] C. B. Gurwitz, *Sequential Quadratic Programming Methods Based on Approximating a Projected Hessian Matrix,* General Books, 2001.