Quaternionic Wigner-Ville distribution of analytical signal in hyperspectral imagery

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***Abstract*—The 2D Quaternionic Fourier Transform (QFT), applied to a real 2D image, produces an invertible quaternionic spectrum. If we conserve uniquely the first quadrant of this spectrum, it is possible, after inverse transformation, to obtain, not the original image, but a 2D quaternion image, which generalize in 2D the classical notion of 1D analytical image.**

From this quaternion image, we compute the corresponding correlation product, then, by applying the direct QFT, we obtain the 4D Wigner-Ville distribution of this analytical signal. With reference to the shift variables χ_1, χ_2 used for the computation of the correlation product, we obtain a local quaternion Wigner-Ville distribution spectrum.

***Keywords*—Analytical; signal; hyperspectral imagery; quaternionic distribution**

I. INTRODUCTION

The most common method of analysis of the frequency content of an n-D real signal is the classical complex Fourier transform. The Fourier transforms have been widely used in signal and image processing. ever since the discovery of the Fast Fourier Transform in 1965 (Cooley-Tukey algorithm) which made the computation of Discrete Fourier Transform feasible using a computer.

Based on the concept of quaternion the quaternion Fourier transform (QFT) has been introduced by Ell (Ref 1) and implemented by Pei (Ref 2) with conventional 2D Fourier transform .

The analytic signal is a complex extension of a 1D signal that is based upon the Hilbert transform; it was introduced to signal theory by Gabor in 1946 (Ref 3).This representation gives access to the instantaneous amplitude and phase. Several attempts to generalize the analytic signal to two dimensions have been reported in the literature, based on the properties of Hilbert and Riesz transforms (Ref 4).

II. CONCEPT OF QUATERNION'S NUMBERS

The Quaternion, discovered by Hamilton (Ref 5) in 1843, also called hyper complex numbers, are the generalization of complex numbers. A complex number has two components: the real and the imaginary part.

The quaternion has four components:

$$q = q_r + q_i \cdot i + q_j \cdot j + q_k \cdot k$$

and *i*, *j*, *k* obeys the rules as below:

$$\begin{aligned} i^2 = j^2 &= k^2 = -1 \\ i \cdot j = k &\quad j \cdot i = -k \\ j \cdot k = i &\quad k \cdot j = -i \\ k \cdot i = j &\quad i \cdot k = -j \end{aligned}$$

We clearly observe that multiplication is associative and distributive, compared to addition, but not commutative.

When the real part is null, the corresponding number is called a pure quaternion. For a quaternion-valued function, we can define its magnitude as follow:

$$|q| = r = \sqrt{q_r^2 + q_i^2 + q_j^2 + q_k^2}$$

like its conjugate as:

$$q* = q_r - q_i i - q_j j - q_k k$$

III. 2D QUATERNIONIC FOURIER TRANSFORM (QFT)

The classic n-D complex Fourier transform is:

$$F_c\,(\mathbf{u}) = \int_{\Re^n} f\,(x) \prod_{i=1}^{i=n} \exp^{-j2\pi u_i x_i} dx^n$$

The QFT is based on the quaternion's concept, the quaternion Fourier Transform of a 2D real signal $f(x, y)$ is defined as:

$$\text{QFT} = F_q(u, v) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-i2\pi ux} f(x, y) e^{-j2\pi vy}\, dx\, dy$$

This QFT, of type 1, is noted two-side. If the input $f(x, y)$ is a quaternion function and not only a real function, we can decompose $f(x, y)$ as:

$$f(x, y) = f_r(x, y) + i.f_i(x, y) + f_j(x, y)j. + f_k(x, y)\text{k}.$$

where $f_r(x, y)$, $f_i(x, y)$, $f_j(x, y)$ and $f_k(x, y)$ are real functions. We obtain

$$F_q(u, v) = F_{rq}(u, v) + iF_{iq}(u, v) + jF_{jq}(u, v) + kF_{kq}(u, v)$$

The QFT is not identical to the 2D Clifford Fourier Transform (Ref 6,7), since the signal f_{CF} is sandwiched between the two exponential functions rather than standing on their left side. However the two transforms are identical for real 2D signals.

The QFT is invertible and its inverse is expressed as:

$$F_q(u,v) = F_{rq}(u,v) + iF_{iq}(u,v) + jF_{jq}(u,v) + kF_{kq}$$

$$f(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i2\pi ux} F_q(u,v) e^{j2\pi vy}\, du\, dv$$

The discrete Quaternion Fourier Transform (DQFT) was introduced by Sangwine and Ell in year 2000 (Ref 7)

This transform has many different expression types .In this paper, we only use the type 1 of DQFT, which has the following expression (direct formulation):

$$F_q(u,v) = \sum_{M=0}^{M-1}\sum_{N=0}^{N-1} e^{-i2\pi\left(\frac{xu}{M}\right)} f(x,y) e^{-i2\pi\left(\frac{yv}{N}\right)}$$

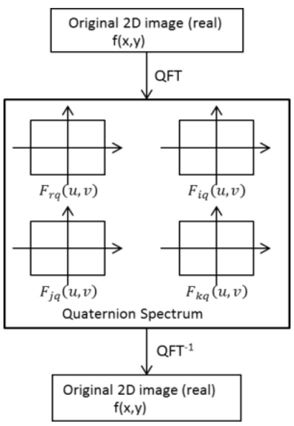


Fig. 1. Quaternion Fourier Transform Original image :

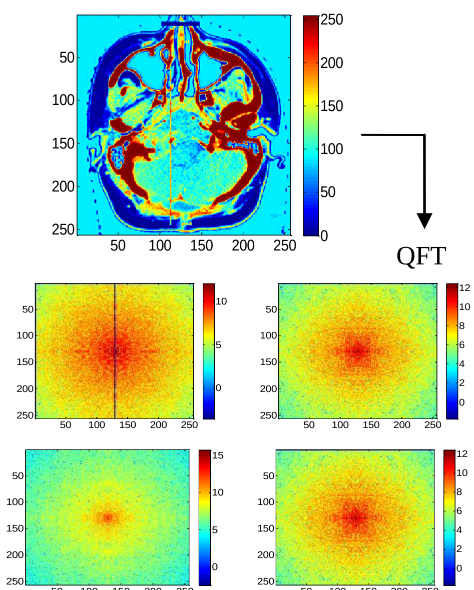


Fig. 2. Quaternionic spectrum

IV. Properties Of Quaternionic Spectral Analysis: Hermitian Symmetry

The quaternion spectrum QFT obeys the rules of the quaternion hermitian symmetry defined by the relations (Ref 8):

$$F_q(-u,v) = \propto_j \cdot F_q(u,v) = -j \cdot F_q \cdot j$$

$$F_q(-u,-v) = \propto_k \cdot F_q(u,v) = -k \cdot F_q \cdot k$$

$$F_q(u,-v) = \propto_i \cdot F_q(u,v) = -i \cdot F_q \cdot i$$

where the functions $\propto_i$, $\propto_j$ and $\propto_k$ are called involutions of F_q

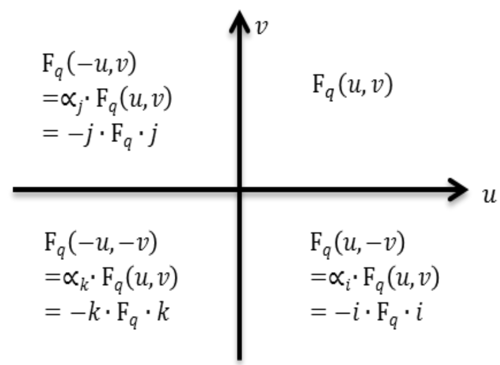


Fig. 3. hermitian properties of QFT

V. Definitions And Obtaining Of Analytical Signal

Given a real n-D signal $f(x)=f(x_1,x_2,\dots\dots x_n)$

The n-D analytical signal with single orthant (the orthant is a half axis in 1-D, a quadrant in 2-D and an octant in3-D)

$$f_{1q}(x) = f(x) \otimes\otimes\otimes \dots \psi^{\delta}(x)$$

The symbol ⊗⊗⊗… denote the n-D convolution.

with: $\psi^{\delta}(x) = \prod_{i=1}^{i=n} \left[\delta x_1 + \frac{e_i}{\pi x_i}\right]$ n-D hypercomplex delta distribution.

$$f_{1q}(x) = f(x) \otimes\otimes \left[\delta x_1 + \frac{e_1}{\pi x_1}\right]\left[\delta x_2 + \frac{e_2}{\pi x_2}\right]$$

$$f_{1q}(x) = f(x) \otimes\otimes \left[\delta x_1\, \delta x_2 + \delta x_1 \frac{e_2}{\pi x_2} + \delta x_2 \frac{e_1}{\pi x_1} + \frac{e_1}{\pi x_1}\frac{e_2}{\pi x_2}\right]$$

The analytical notion can be extended to taking the quaternions in Clifford algebra which contains the elements :

with: $e_1=i$ $e_2=j$ $e_1e_2=k$

$$f_{1q}(x) = f(x_1, x_2) + ih_1 + jh_2 + kh$$

$$h = f(x) \otimes\otimes \frac{1}{\pi x_1}\,\frac{1}{\pi x_2} \quad \text{total Hilbert transform}$$

$$h_1 = f(x) \otimes\otimes \delta x_2 \frac{1}{\pi x_1} \quad \text{and}$$

$$h_2 = f(x) \otimes\otimes \delta x_1 \frac{1}{\pi x_2} \quad \text{partial Hilbert transforms}$$

In the spatial plane $f_{1q}(x)$ is a quaternion which the four images are f(x), h_1, h_2 and h, represented in the fig 5.

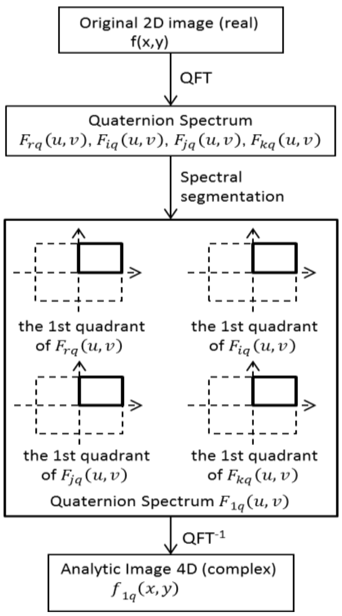
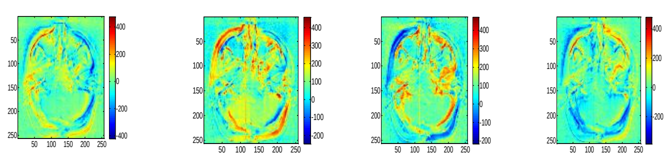


Fig. 4. Obtaining analytic image



Real i j k

Fig. 5. Analytical images in the space plan

VI. Quaternion Wigner –Ville Distribution

The quaternion Wigner-Ville distribution (Ref 9,10) is defined by the QFT of the analytical signal f_{1q} (x, y) with reference to shift variables (χ_1, χ_2) of quaternionic correlation product ρ

$$\rho(x, y, \chi_1, \chi_2) = f_{1q}(x + \chi_1/2, y + \chi_2/2) \cdot f_{1q}{}^{*}(x - \chi_1/2, y - \chi_2/2)$$

$$f_{1q}(x, y) = f(x, y) + i.h_1(x, y) + j.h_2(x, y) + k.h(x, y)$$

we obtain:

$$W_q(x, y, u, v) = \iint e^{-i2\pi u\chi_1}\ \rho(x, y, \chi_1, \chi_2)\ e^{-j2\pi v\chi_2}\, d\chi_1\, d\chi_2$$

$$W_q(x, y, u, v) = W_{qr}(x, y, u, v) + \mathrm{i}.W_{qi}(x, y, u, v)$$

$$+k.W_{qk}(x, y, u, v) + j.W_{qj}(x, y, u, v)$$

$$\rho\ (x,y,u,v) = A_q + iB_q + jC_q + kD_q$$

$$A_q = f^+f^- + h_1^+h_1^- + h_2^+h_2^- + h^+h^-$$
$$B_q = h^+h_2^- - f^+h_1^- + h_1^+f^- - h_2^+h^-$$
$$C_q = -f^+h_2^- - h^+h_1^- + h_1^+h^- + h_2^+f^-$$

$$D_q = h^+f^- - f^+h^- + h_2^+h_1^- - h_1^+h_2^-$$

$$f^+ = f(x+0{,}5\chi_1),(y+0{,}5\chi_2)$$
$$f^- = f(x-0{,}5\chi_1),(y-0{,}5\chi_2)$$

with the same for the other functions h_1, h_2 and h.

$$W_q(x, y, u, v) = \iint e^{-i2\pi u\chi_1}(\,Aq + iBq + jCq + kDq)e^{-j2\pi v\chi_2}\, d\chi_1\, d\chi_2$$

$$W_q(x, y, u, v) = \iint e^{-i\alpha 1}(\,Aq + iBq + jCq + kDq)e^{-j\alpha 2}\, d\chi_1\, d\chi_2$$

If $\alpha_1 = 2\pi u\chi_1$ and $\alpha_2 = 2\pi v\chi_2$

$e^{-i\alpha 1} = \cos\alpha_1 - i\sin\alpha_1$ and $e^{-j\alpha 2} = \cos\alpha_2 - i\sin\alpha_2$

In the spectral space, the quaternionic Wigner distribution is a 4-D quaternion function :

$$W_{qr}(x,y,u,v) = \iint (Aq\cos\alpha 1\ \cos\alpha 2 + Bq\sin\alpha 1\cos\alpha 2 + Cq\cos\alpha 1\sin\alpha 2 + Dq\sin\alpha 1\sin\alpha 2)\, d\chi_1\, d\chi_2$$

$$W_{qi}(x,y,u,v) = \iint (Bq\cos\alpha 1\ \cos\alpha 2 - Aq\sin\alpha 1\cos\alpha 2 + Dq\cos\alpha 1\sin\alpha 2 - Cq\sin\alpha 1\sin\alpha 2)\, d\chi_1\, d\chi_2$$

$$W_{qj}(x,y,u,v) = \iint (Cq\cos\alpha 1\ \cos\alpha 2 + Dq\sin\alpha 1\cos\alpha 2 - Aq\cos\alpha 1\sin\alpha 2 - Bq\sin\alpha 1\sin\alpha 2)\, d\chi_1\, d\chi_2$$

$$W_{qk}(x,y,u,v) = \iint (Dq\cos\alpha 1\ \cos\alpha 2 - Cq\sin\alpha 1\cos\alpha 2 - Bq\cos\alpha 1\sin\alpha 2 + Aq\sin\alpha 1\sin\alpha 2)\, d\chi_1\, d\chi_2$$

(this equation : $W_q = W_{qr} + iW_{qi} + jW_{qj} + kW_{qk}$ holds for monogenic signals changing the subscripts q to M)

In the particular case u=0 and v=0

$$W_{qr}(x,y,0,0) = \iint Aq\, d\chi_1\, d\chi_2$$

$$W_{qr}(x,y) = \iint [f^+f^- + h_1^+h_1^- + h_2^+h_2^- + h^+h^-]\ \ d\chi_1\, d\chi_2$$

The local spectrum, centered on u = 0 and v = 0, is the sum of four spectra obtained from four correlation products, corresponding to the four components of the quaternionic analytic signal.

VII. Generalization For 3-D Real Signal.

For n=3,the hypercomplex delta distribution ψ^{δ} (x) become:

$$\psi^{\delta}(x_1,x_2,x_3) = \delta x_1\, \delta x_2\, \delta x_3 + \left[\delta x_1 + \frac{e_1}{\pi x_1}\right]\left[\delta x_2 + \frac{e_2}{\pi x_2}\right]\left[\delta x_3 + \frac{e_3}{\pi x_3}\right]$$

and with the rules of Cayley-Dickson (Ref 9,11)

$$\psi^{\delta}(x_1,x_2,x_3) = \delta x_1\, \delta x_2\, \delta x_3 + e_1 \frac{\delta x_2 \delta x_3}{\pi x_1} + e_2 \frac{\delta x_1 \delta x_3}{\pi x_2} + e_3 \frac{\delta x_3}{\pi^2 x_1 x_3} + e_4 \frac{\delta x_1 \delta x_2}{\pi x_3} + e_5 \frac{\delta x_2}{\pi^2 x_1 x_3} + e_6 \frac{\delta x_1}{\pi^2 x_1 x_3} + e_7 \frac{1}{\pi^3 x_1 x_2 x_3}$$

which is a octonion structure

$$f_{CD}(x_1,x_2,x_3) = f(x_1,x_2,x_3) \otimes\otimes\otimes \Big[\, \delta x_1\, \delta x_2\, \delta x_3 + e_1 \frac{\delta x_2 \delta x_3}{\pi x_1} + e_2 \frac{\delta x_1 \delta x_3}{\pi x_2} + e_3 \frac{\delta x_3}{\pi^2 x_1 x_3} + e_4 \frac{\delta x_1 \delta x_2}{\pi x_3} + e_5 \frac{\delta x_2}{\pi^2 x_1 x_3} + e_6 \frac{\delta x_1}{\pi^2 x_1 x_3} + e_7 \frac{1}{\pi^3 x_1 x_2 x_3} \Big]$$

which corresponds to 8 images in the spectral plan: one real and seven imaginaries.

The process to obtain the Wigner-Ville distribution is the same as in the previous case n=2, but the computational complexity is increased by the passage, for the analytic signal, of an quaternionic structure to an octonion.

VIII. Conclusion

The use of the Fourier quaternionic transformation, associated at the convolution with the n-D delta distribution ,allows to obtain, from real image signal, a spatially hyper complex representation of the analytical signal (quaternion in 2D and octonion in 3D).From this approach , we can generalize the concept of the Wigner Ville distribution and obtain an analytical. tool with both a frequency selectivity and spatial localization.

Interesting applications can be envisaged in imagery, especially for the segmentation problems or in texture analysis (Ref 12).

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