Dynamic Modelling of Flexible Manipulator based on a Large Number of Finite Elements

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***Abstract*—This paper focuses on the issue of tracking the trajectory of a flexible arm. The purpose is to ensure the flexible arm follows the desired path in the joint space. To achieve our objective, we have three problems to solve: modeling, control, and trajectory planning. As in the case of rigid robots, the Euler-Lagrange formulation remains valid with the exception of dividing the flexible arm into a finite number of elements to model the deformation. The iterative learning control scheme can be used to achieve perfect tracking throughout the movement period, a sufficient condition based on the bounded real lemma that guarantees the convergence error between iteration is given. All the results are presented in terms of linear matrix inequalities synthesis (LMIs).**

Keywords—Single-link flexible arm; finite element; trajectory tracking; iterative learning control; linear matrix inequality; bounded real lemma

I. Introduction

Robotics, a discipline that has emerged over a few decades, owes its present development to the pooling and coordination of research results in several sciences. When designing a robot, problems with mechanics, electronics, automation, computer science, and language theory must be regulated. In the manufacturing industry, difficult and repetitive tasks are entrusted to articulated mechanical systems.

Among the rigid and flexible manipulator types, attention is focused more recently towards flexible manipulators. This is due to various advantages such manipulators offer in comparison to their rigid counterparts [1], they require less material, are lighter in weight, have higher manipulation speed, lower power consumption, require smaller actuators, are more maneuverable and transportable, and are safer to operate due to reduced inertia. However, the control of flexible manipulators is more complicated by the complexity of the system dynamics, the residual vibrations, and, especially, the uncertainties of the final response of the end effector's extensibility due to deformation. Therefore, there is a need to continuously improve the mathematical models and control methods in order to fulfill conflicting requirements.

Several research works focus on the modeling of the flexible arm, such as [2]-[4]. As it is introduced by [5], [6], the finite element method (FE) has proven to be one of the best methods to obtain a good description of the dynamics of the system because it is able to function with irregularities in the structure of the arm with managing mixed boundary conditions and it makes allowance for interaction between the gross motion and the flexible dynamics of the manipulator, which is not possible with use of methods based on frequency domain analysis. Unfortunately, the major disadvantage of the FE method is the computational complexity and the difficult software coding involved, this is the primary reason several works such as [7], [8] have been limited to two or three finite elements. Given that confidence in the accuracy of the model is crucial for utilization in subsequent investigations and development of control strategies, a dynamic model of a flexible arm including hub inertia and payload has been developed using four finite elements.

Iterative learning control (ILC) is a preferable technique when it comes to dealing with robot manipulators because they execute the same task repeatedly over a finite time interval [9]. Iterative learning control has been shown to be effective in improving tracking performance of repetitive tasks and is widely used in motion control systems [10], [11]. Due to the fact that the majority of tasks realized by the flexible arm are repetitive like welding or picking and placing, the tracking problems in the joint-space can be treated under ILC framework. The main idea of ILC is to improve the performance from one iteration to another in the sense that the tracking error is sequentially reduced by using information from previous executions of the task.

The obtained formulation when applying iterative learning control (ILC) control is transformed into a synthesis problem of a repetitive system [12]. Using the benefits of the bounded real lemma (BRL) from the Robust Control Theory the output error between the desired and the actual trajectory is monotonically convergent (MC) to zero with the progress of the learning process. The stability analysis is presented and the convergence conditions for the system are expressed by LMIs which can determine the switching learning gains. For the designs and simulations, the software MATLAB was employed.

The remainder of this paper is organized into five sections. Section 2 is interested in the dynamic modeling of this flexible arm. Section 3 is dedicated to convergence analysis using ILC control, Monotonically Convergent conditions presented in Section 4. Simulation results of the trajectory tracking and the torques applied to the joint are presented in Section 5. Finally, concluding remarks are given in Section 6.

Notation used in the paper is standard. In general capital letters denote matrices. For two symmetric matrices, A^T

Denotes the transpose of A, $diag\,(x, y, ..)$ denotes the diagonal matrix obtained from vectors or matrices x, y,... .Identity and null matrices will be denoted respectively by I and 0. Furthermore, in the case of partitioned symmetric matrices, the symbol $*$ denotes generically each of its symmetric blocks.

II. Dynamic Modeling

In this section we developed a dynamic model for one-link flexible arm using the FE method with four finite elements, a description of the system is given with the assumptions utilized in modeling, a MATLAB code is developed based on the theory of the finite element method (FEM).

The main idea of FEM is to treat complex structures as a finite assembly of discrete elements with continuous structures; each element has its own kinetic and potential energy to consider in determining the total system energy and applying the Lagrange formalism.

A. Assumptions

Our system, the flexible arm, is pivotally connected to the support (the base) at the hub; this rotary linkage is performed by a direct current (DC) motor. A schematic representation of the single-link flexible manipulator system is shown in Fig. 1, with E, I, Im, A, ρ, l, τ and m, respectively representing Young's modulus, the second moment of area, the hub inertia moment, the section, the mass density, the length, the rotation angle of the arm relative to the hub and the mass of the effector.

To apply the FE, we begin by dividing the beam into a finite number of successive elements of equal length; the points of intersection between its elements are defined as articulations. Following this, we calculate the potential energy and the kinetic energy for each element to ascertain the total energy of the whole system. The determination of the kinetic and potential energy of the whole system (the beam divided into finite element, the effector and the motorized articulation) is essential in applying the Lagrange Formalism which permits us to attain the dynamic model.

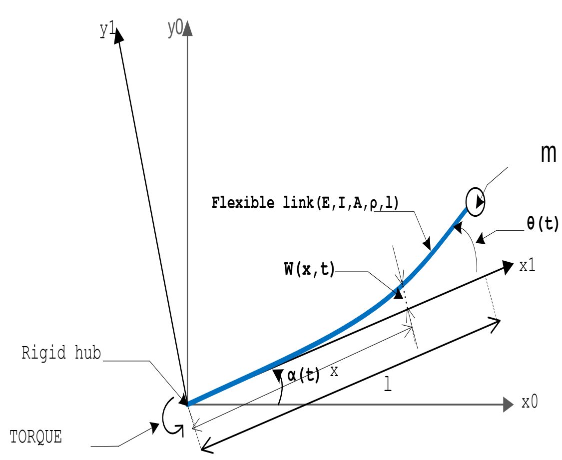

Fig. 1. Flexible arm scheme.

We assumed the following hypothesis:

A1: The depth of the flexible arm is much smaller than its length.

A2: The effect of the axial force and the rotational inertia are negligible.

B. Finite Element Method

For an angular displacement α and an elastic deflection w, the total displacement $y(x,t)$ of a point along the manipulator at distance from the hub is donated by:

$$y(x,t) = \alpha\,(t) + w(x,t) \tag{1}$$

As it defined in [1] the total displacement can be also described by:

$$y(x,t) = \sum_{i=1}^{4} N_i(x)u_i(t) \tag{2}$$

Where, $u(t)$ and $N(x)$ are the nodal displacement and shape function correspondingly. In this work, the shape functions are Hermit cubic functions as it is defined in [2]. Using (2) we can obtain the expression of the energy for one element, where T is the kinetic energy for one element and V is the potential energy for one element.

$$T(t) = \frac{1}{2}\int_0^L \rho A\left(\frac{dy(x,t)}{dt}\right)^2 dx \tag{3}$$

$$V(t) = \frac{1}{2}\int_0^L EI\left(\frac{d^2y(x,t)}{dx^2}\right)^2 dx \tag{4}$$

Based on the work of [4], [8], we can put the energy of the element in a matrix form and this is crucial for coding in MATLAB, the total kinetic energy of the flexible arm without considering the effector (mass m) will be then the sum of n elementary kinetic energies and the total potential energy will be the sum of n elementary energies:

$$Ec = \frac{1}{2}\sum_{i=1}^{n} \dot{q}_i^{\,T} z\, \dot{q}_i \tag{5}$$

$$Ep = \frac{1}{2}\sum_{i=1}^{n} \dot{q}_i^{\,T} s\, \dot{q}_i \tag{6}$$

With Ep, Ec, z, s, q being the total potential energy, the total kinetic energy, the mass matrix for one element, the stiffness matrix for one element and $q_i = [\alpha\ w_i\ \theta_i\ w_{i+1}\ \theta_{i+1}]^T$ being the vector of joint variables for each element.

Applying the Lagrange formulation defined in (7), we obtain the system mass and system stiffness matrices, Z and S. These matrices correspond to the flexible arm system (excluding the effector).

$$\frac{d}{dt}\frac{dEc}{d\dot{q}_i} - \frac{dEc}{dq_i} + \frac{dEp}{dq_i} = 0 \quad (7)$$

Using these matrices we define the potential energy and the kinetic energy for the whole system (the beam divided into finite element, the effector and the motorized articulation) .

$$Egc = \frac{1}{2} Im\,\dot{\alpha}^2 + \frac{1}{2}\left(\dot{q} + L\dot{\alpha}\right)^T Z\left(\dot{q} + L\,\dot{\alpha}\right) \quad (8)$$

$$Egp = \frac{1}{2}\, q^T\ S\ q \quad (9)$$

Note that Egp stands for the potential energy of the entire system, Egc the kinetic energy of the entire system, and $L^T = [L_1 L_2 ......L_n]^T$ the vector of the lengths. L_i separating the node i at the origin of the frame before the flexion of the arm. After determining Egp and Egc , application of the Lagrange formalism is required in order to attain the dynamic model (10).

$$\frac{d}{dt}\frac{dEgc}{d\dot{q}_i} - \frac{dEgc}{dq_i} + \frac{dEgp}{dq_i} = \tau_i \quad (10)$$

With τ_i is the Torques or forces applied to the joint.

The dynamic model acquired is characterized by (11) where q is the vector of the joint variables, M is the global mass matrix and K is the global stiffness matrix.

$$\tau = M\ \ddot{q} + K\ q \quad (11)$$

For four finite elements the two matrices M and k are squares of size 9 and given by (17) and (18).

III. Convergence Analysis using ILC Control

A. Problem Formulation

The dynamic equations described in Section 2 can be presented in a state-space form of a repetitive system.

$$\begin{cases} \dot{x}_k(t) = Ax_k(t) + Bu_k(t) \\ y_k(t) = Cx_k(t) + Du_k(t) \end{cases},\quad x_k(0) = 0, t \geq 0 \quad (12)$$

Where the state-space matrices are:

$$A = \begin{pmatrix} 0_{(9*9)} & I_{(9*9)} \\ M^{-1}K & 0_{(9*9)} \end{pmatrix} \quad (13)$$

$$B = \begin{pmatrix} 0_{(9*1)} \\ M^{-1} \end{pmatrix} \quad (14)$$

$$C = (1 \quad 0_{(1*17)}) \quad (15)$$

$$D = (0) \quad (16)$$

$x_k \in \mathbb{R}^n$ is the state vector given by $x_k{}^T = [\alpha\ w_1\ \theta_1\ w_2\ \theta_2\ w_3\ \theta_3\ w_4\ \theta_4\ \dot{\alpha}\ \dot{w}_1\ \dot{\theta}_1\ \dot{w}_2\ \dot{\theta}_2\ \dot{w}_3\ \dot{\theta}_3\ \dot{w}_4\ \dot{\theta}_4]^T$ and $u_k \in \mathbb{R}^n$ is the input control given by $u_k = [\tau\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0]$

$$M(1,1) = \frac{A L_1 \rho\ (7458 L_1 + 15475\ L_2)}{50000}, M(1,2) = \frac{A L_1 \rho\ (15567 L_1 + 12344\ L_2)}{50000}$$

$$M(1,3) = \frac{A L_1 \rho\ (8935 L_1 + 23501\ L_2)}{50000}, M(1,4) = \frac{A L_1 \rho\ (44098 L_2 + 36827\ L_3)}{50000}$$

$$M(1,5) = \frac{A L_1 \rho\ (37382 L_2 + 96605\ L_3)}{10000},\ M(1,6) = \frac{499 A L_1 \rho\ L_3}{12500}$$

$$M(1,7) = \frac{-(131\ A L_1 \rho\ L_3)}{3125}, M(1,8) = 0, M(1,9) = 8\ L_1 m$$

$$M(2,1) = \frac{A\rho\ (15742 L_1 + 2570\ L_2)}{50000},\ M(2,2) = \frac{A\rho\ (11621 L_1 + 10239\ L_2)}{50000}$$

$$M(2,3) = \frac{A\rho\ (3946 L_1 + 3055\ L_2)}{100000},\ M(2,4) = \frac{-69 A\ L_2 \rho}{5000}$$

$$M(2,5) = \frac{7 A\rho\ L_2}{1250},\ M(2,6) = 0, M(2,7) = 0, M(2,8) = 0, M(2,9) = 0$$

$$M(3,1) = 0, M(3,2) = \frac{A\rho\ (3946 L_1 + 3055\ L_2)}{10^5}$$

$$M(3,3) = \frac{A\rho\ (349 L_1 + 1093\ L_2)}{5000}, M(3,4) = \frac{A L_2 \rho\, 2033}{100000}$$

$$M(3,5) = \frac{-367 A\rho\ L_2}{25000}, M(3,6) = 0, M(3,7) = 0, M(3,8) = 0, M(3,9) = 0$$

$$M(4,1) = 0, M(4,2) = \frac{-69 A L_2\ \rho}{5000}, M(4,3) = \frac{A\rho\, 2033 L_2}{100000}$$

$$M(4,4) = \frac{A\rho(11621 L_2 + 10239 L_3)}{50000}, M(4,5) = \frac{A\rho\,(3946 L_2 + 3055 L_3)}{100000}$$

$$M(4,6) = \frac{-69 A L_3\ \rho}{5000}, M(4,7) = \frac{7 A L_3\ \rho}{1250}, M(4,8) = 0, M(4,9) = 0$$

$$M(5,1) = 0, M(5,2) = \frac{7 A L_2\ \rho}{1250}, M(5,3) = \frac{-367 A\rho L_2}{25000}$$

$$M(5,4) = \frac{A\rho(3946 L_2 + 3055 L_3)}{100000}, M(5,5) = \frac{A\rho\,(394 L_2 + 1093 L_3)}{5000}$$

$$M(5,6) = \frac{2033 A L_3\ \rho}{100000}, M(5,7) = \frac{-367 A L_3\ \rho}{25000}, M(5,8) = 0, M(5,9) = 0$$

$$M(6,1) = 0, M(6,2) = 0, M(6,3) = \frac{154018 A\rho\ L_3}{5000}, M(6,4) = \frac{-69 A\rho\ L_3}{5000}$$

$$M(6,5) = \frac{2033 A\rho\ L_3}{100000}, M(6,6) = \frac{A\rho\,(11621 L_3 + 10293 L_4)}{50000}$$

$$M(6,7) = \frac{A\rho\,(3946 L_3 + 3055 L_4)}{100000}, M(6,8) = \frac{-69 A\rho L_4}{5000}$$

$$M(6,9) = \frac{7 A\rho\ L_4}{1250}, M(7,1) = 0, M(7,2) = 0, M(7,3) = 0, M(7,4) = \frac{7 A\rho\ L_3}{1250}$$

$$M(7,5) = \frac{-367 A\rho\ L_3}{25000}, M(7,6) = \frac{A\rho\,(3946 L_3 + 3055 L_4)}{100000}$$

$$M(7,7) = \frac{A\rho\,(349 L_3 + 1093 L_4)}{5000}, M(7,8) = \frac{2033 A\rho L_4}{100000}, M(7,9) = \frac{-367 A\rho\ L_4}{25000}$$

$$M(8,1) = 0, M(8,2) = 0, M(8,3) = 0, M(8,4) = 0, M(8,5) = 0$$

$$M(8,6) = \frac{-69 A\rho\ L_4}{5000}, M(8,7) = \frac{2033 A\rho L_4}{100000}, M(8,8) = \frac{11621 A\rho\ L_4}{50000}$$

$$M(8,9) = \frac{1973 A\rho L_4}{50000}, M(9,1) = \frac{4730 A\rho L_4}{5000}, M(9,2) = \frac{2254 A\rho L_4}{1250}$$

$$M(9,3) = 0, M(9,4) = 0, M(9,5) = 0, M(9,6) = \frac{7 A\rho\ L_4}{1250}$$

$$M(9,7) = \frac{-367 A\rho L_4}{25000}, M(9,8) = \frac{1973 A\rho L_4}{50000}, M(9,9) = 2m + \frac{349 A\rho L_4}{5000} \quad (17)$$

$$K(1,1)=0, K(1,2)=0, K(1,3)=0, K(1,4)=0, K(1,5)=0, K(1,6)=0$$
$$K(1,7)=0, K(1,8)=0, K(1,9)=0$$
$$K(2,1)=0, K(2,2)=\frac{2\,E\,I\,(58174\,L_1^3+30526\,L_2^3)}{97L_1^3\,L_2^3}$$
$$K(2,3)=\frac{-2\,E\,I\,(36672\,L_1^3+11328\,L_2^3)}{97L_2^3L_1^3}, K(2,4)=\frac{27648\,E\,I}{97L_2^3}$$
$$K(2,5)=\frac{-4608\,E\,I}{97L_2^3}, K(2,6)=0, K(2,7)=0, K(2,8)=0, K(2,9)=0$$
$$K(3,1)=0, K(3,2)=\frac{-2\,E\,I\,(36672\,L_1^3+11328\,L_2^3)}{97L_1^3L_2^3}$$
$$K(3,3)=\frac{2\,E\,I\,(44350\,L_1^3+4992\,L_2^3)}{97\,L_2^3L_1^3}, K(3,4)=\frac{-59520E\,I}{97L_2^3}$$
$$K(3,5)=\frac{16128\,E\,I}{97L_2^3}, K(3,6)=0, K(3,7)=0, K(3,8)=0, K(3,9)=0$$
$$K(4,1)=0, K(4,2)=\frac{27648\,E\,I}{97L_2^3}, K(4,3)=\frac{-59520\,E\,I}{97L_2^3}$$
$$(4,4)=\frac{2\,E\,I\,(58174\,L_2^3+30526\,L_3^3)}{97L_2^3L_3^3}$$
$$K(4,5)=\frac{-2\,E\,I\,(36672\,L_2^3+11328\,L_3^3)}{97L_2^3L_3^3}, K(4,6)=\frac{27648\,E\,I}{97L_3^3}$$
$$K(4,7)=\frac{-4608\,E\,I}{97L_3^3}, K(4,8)=0, K(4,9)=0$$
$$K(5,1)=0, K(5,2)=\frac{-4608\,E\,I}{97L_3^3}, K(5,3)=\frac{16128\,E\,I}{97L_2^3}$$
$$(5,4)=\frac{-2\,E\,I\,(36672\,L_2^3+11328\,L_3^3)}{97L_2^3L_3^3}$$
$$K(5,5)=\frac{2\,E\,I\,(44350\,L_2^3+4992\,L_3^3)}{97L_2^3L_3^3}, K(5,6)=\frac{-59520\,E\,I}{97L_3^3}$$
$$K(5,7)=\frac{16128\,E\,I}{97L_3^3}, K(5,8)=0, K(5,9)=0, K(6,1)=0$$
$$K(6,2)=0, K(6,3)=0, K(6,4)=\frac{27648\,E\,I}{97L_3^3}$$
$$K(6,5)=\frac{-59520\,E\,I}{97L_3^3}, K(6,6)=\frac{2\,E\,I\,(58174\,L_3^3+30526\,L_4^3)}{97L_3^3L_4^3}$$
$$K(6,7)=\frac{-2\,E\,I\,(36672\,L_3^3+11328\,L_4^3)}{97L_3^3L_4^3}$$
$$K(6,8)=\frac{27648\,E\,I}{97L_4^3}, K(6,9)=\frac{-4608\,E\,I}{97L_4^3}, K(7,1)=0$$
$$K(7,2)=0, K(7,3)=0, K(7,4)=\frac{27648\,E\,I}{97L_3^3}$$
$$K(7,5)=\frac{-59520\,E\,I}{97L_3^3}, K(7,6)=\frac{2\,E\,I\,(58174\,L_3^3+30526\,L_4^3)}{97L_3^3L_4^3}$$
$$K(7,7)=\frac{-2\,E\,I\,(36672\,L_3^3+11328\,L_4^3)}{97L_3^3L_4^3}$$
$$K(7,8)=\frac{27648\,E\,I}{97L_4^3}, K(7,9)=\frac{-4608\,E\,I}{97L_4^3}$$
$$K(8,1)=0, K(8,2)=0, K(8,3)=0, K(8,4)=0, K(8,5)=0$$
$$K(8,6)=\frac{27648\,E\,I}{97L_4^3}, K(8,7)=\frac{-59520\,E\,I}{97L_4^3}$$
$$K(8,8)=\frac{27648\,E\,I}{97L_4^3}, K(8,9)=\frac{-22656\,E\,I}{97L_4^3}, K(8,1)=0$$
$$K(8,2)=0, K(8,3)=0K(8,4)=0, K(8,5)=0$$
$$K(8,6)=\frac{27648\,E\,I}{97L_4^3}, K(8,7)=\frac{-59520\,E\,I}{97L_4^3}$$
$$K(8,8)=\frac{61052\,E\,I}{97L_4^3}, K(8,9)=\frac{-22656\,E\,I}{97L_4^3}, K(9,1)=0$$
$$K(9,2)=0, K(9,3)=0, K(9,4)=0, K(9,5)=0$$
$$K(9,6)=\frac{-4608\,E\,I}{97L_4^3}, K(9,7)=\frac{16128\,E\,I}{97L_4^3}$$
$$K(9,8)=\frac{22656\,E\,I}{97L_4^3}, K(9,9)=\frac{9984\,E\,I}{97L_4^3} \qquad (18)$$

The objective is to apply an ILC control to the system (12) to ensure stability and the trajectory tracking of desired trajectory $y_d(t)$ based on the following hypothesis.

B1 : The desired trajectory $y_d(t)$ is iteration invariant.

B2 : Every operation begins at an identical initial condition $x_k(0)=0$.

In this section, the formulated problem is solved by using the ILC control described by the following form:

$$u_{k+1}(t)=\beta u_k(t)+K_1\dot{\eta}_{k+1}(t+1)+K_2\dot{e}_k(t) \qquad (19)$$

$$\eta_{k+1}(t+1)=\int_0^t\left(\dot{x}_{k+1}(t)-\beta\dot{x}_k(t)\right)dt \qquad (20)$$

Where, $e_k(t)=y_d(t)-y_k(t)$ is the output tracking error, $\dot{\eta}_{k+1}(t)$ denotes the state vector between two iteration, K_1, K_2 are learning gains with appropriately dimensioned matrices to be designed, and β a positive scalar. Replacing (12) in (19) and respecting (20), we obtain the state space representation of the closed loop system.

$$\begin{bmatrix}\dot{\eta}_{k+1}(t)\\ e_{k+1}(t)\end{bmatrix}=\begin{bmatrix}(A+BK_1) & BK_2\\ -C(A+BK_1) & (\beta-CBK_2)\end{bmatrix}\begin{bmatrix}\eta_{k+1}(t)\\ e_k(t)\end{bmatrix} \qquad (21)$$

The state space representation (21) includes two independent dynamic processes: one along the time axis t and the other along the iteration axis k. As it is specified [13] we have G_{bf} the transfer from $e_k(t)$ to $e_{k+1}(t)$.

$$G_{bf}=\begin{bmatrix}A_{bf} & B_{bf}\\ C_{bf} & D_{bf}\end{bmatrix}=\begin{bmatrix}\left(A_{k+1}+BK_1\right) & BK_2\\ -C\left(A_{k+1}+BK_1\right) & \left(\beta-CBK_2\right)\end{bmatrix} \qquad (22)$$

The design objective is to minimize the H_∞-norm of the closed-loop transfer function G_{bf} for $e_k(t)$ to $e_{k+1}(t)$ that is to say:

$$\left\|G_{e_{k+1/e_k}}\right\|_\infty<\gamma \qquad (23)$$

Remark1: In the remainder of this paper, it will be shown that the ILC law can help the tracking error l_2 - norm monotonically converges to zero along the iteration direction.

B. *Tracking Error Convergence*

Due to the problem formulation producing the system (12) with ILC control (19) and respecting hypothesis B1) and B2), we found the appropriate learning gains K_1, K_2 such that the monotonic convergence in (22) is achieved, and the output error $e_{k+1}(t)$ converges to zero as $k\to\infty$, for $t\geq 0$, $k\in IN$.

Definition 1. [14] given the system (12) and ILC controller (19), with B1) and B2), then, (22) is monotonically convergent in $e_k(t)$ if there exists $0<\gamma<1$, $\forall k\in IN$ such that

$$\|e_{k+1}(t)\|_2 < \gamma \|e_k(t)\|_2 \tag{24}$$

Where, $e_k(t)$ the output error of system in iteration k, and $e_{k+1}(t)$ the output error of system in iteration $k+1$.The norm $\|e_k(t)\|_2$ is defined by:

$$\|e_k(t)\|_2 = \sqrt{\left(\int_0^t e_k^{\,T}(t)e_k(t)\right)}dt \tag{25}$$

IV. H_∞ Monotonically Convergent Condition

In this section, a sufficient MC condition for the new system (22) is introduced in terms of LMIs.

Theorem 1: For given a scalar $0<\gamma<1$, the system (12) with ILC control law (19), then (23) is convergent in $e_k(t)$, if there is symmetric positive matrix X, and the matrix N_1, K_1, K_2 with appropriate dimensions, and a scalar $\beta\in[0,1]$, such that the following LMI conditions are satisfied:

$$\begin{bmatrix} AX+BN_1+sym(...) & * & * \\ (BK_2)^T & -I & * \\ (-CAX-CBN_1) & (\beta-CBK_2) & -\gamma^2 I \end{bmatrix} < 0 \tag{26}$$

In this case K_1 are given by $K_1 = N_1(X)^{-1}$.

Proof: First, we consider the increased system (24), if is $\dot{\eta}_{k+1}(t)$ the input signal and $e_{k+1}(t)$ is the output signal. The iterative learning control law can guarantee the monotonic convergence of the output error between the desired output and the actual output for the entire time interval through the iterative learning process.

Applying the BRL [14] to (23), for sufficient condition for the convergence of $\|G_{bf}\|_\infty < \gamma$ it is necessary to determine the positive definite $P>0$ for the following inequality:

$$\begin{bmatrix} (A+BK_1)^T P + sym(..) & * & * \\ (BK_2)^T P & -I & * \\ -C(A+BK_1)^T & (\beta-CBK_2) & -\gamma^2 I \end{bmatrix} < 0 \tag{27}$$

Given the matrix $X=P^{-1}$, and multiplying the condition (26) twice, once by $diag(X,I,I)$ (on the left) and, the other, by the transpose of $diag(X,I,I)$ (on the right), we have the condition (25). The proof is complete.

V. Results and Analysis

In this section, we apply a motion profile as an input to the system (22), as it is proved in [15] the fifth-polynomial profile can provide smooth movement to the single-link flexible arm. This movement is called a minimum-jerk movement, jerk meaning the derivative of acceleration. A motion profile with limited jerk is remarkably similar to the movement of human joints and it reduces the excitation of natural modes.

For the simulation we took:

$$Im = 0.004\,kg.m^2,\ A=0.00002m^2,\ \rho = 3250kg.m^{-2},\ l=0.6m$$
$$I=1.6710^{-12}m^4\quad E=193Pa\quad and\ \ m=0.2kg$$

For illustration purposes, the resulting gains obtained by applying Theorem 1 guarantee the monotonic convergence of error as listed below:

$$K_1 = [0\ -3.6264\ \ 2.2281\ -0.8060\ 0.2299\ -0.0006\ \ -0.0002\ -0.0017\ 0\ -0.001\ -0.002\ 0\ 0\ 0\ 0\ 0\ 0\ 0] \tag{28}$$

$$K_2 = [9.1297\ 10^{-11}] \tag{29}$$

Fig. 2 shows the time and iteration evolution of the output error $e_k(t) = y_d(t) - y_k(t)$. As shown in this figure, the tracking error converges to zero along the iteration and becomes more accurate as the iteration number increases. For error cancellation between the motion profile and the system response we needed ten iterations, the number of iterations is due generally to the size of the system, increasing the number of finite elements during modeling causes the increase of the size of the system since each element adds state variables.

Fig. 3 shows the time evolution of the reference trajectory and the output, after the ninth iterative the system converges completely to the input, this convergence requires a time of 0.45 seconds which shows the rapidity of the control law in the time domain in contrast with the iteration domain, this total convergence is nearly impossible with conventional control methods like the inverse dynamics control.

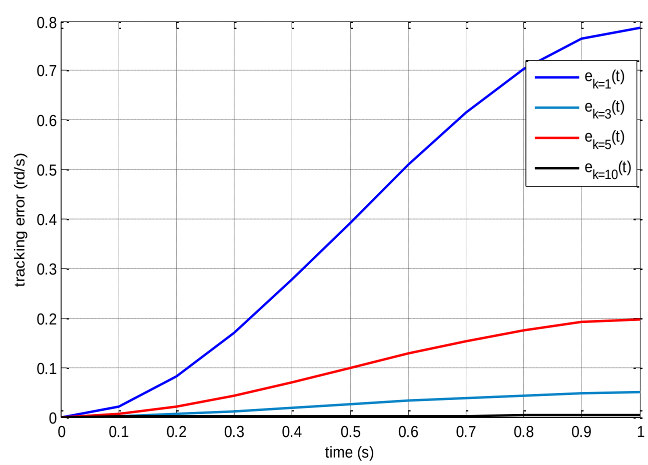

Fig. 2. Evolution of tracking error between iteration.

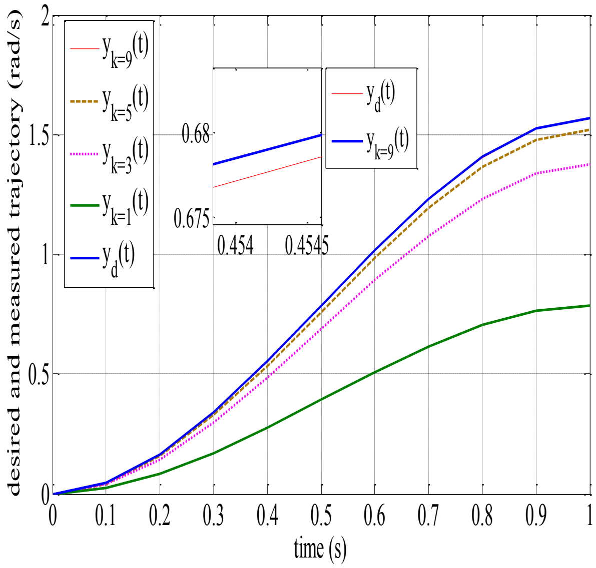

Fig. 3. Desired and measured output evolution.

VI. Conclusion

In this paper, the complication of tracking trajectory for a smooth motion profile of a single-link flexible arm has been investigated. The dynamic model of the manipulator is obtained by using the finite elements method with four finite elements to ensure the accuracy of the model. The system is considered as a repetitive system and an ILC controller is employed to ensure the cancellation of error. Sufficient conditions for the existence of such controller are formulated in terms of a set of LMIs. A numerical example is displayed to illustrate the effectiveness of the proposed methods. In conclusion, the result depicted is an important step towards analyzing physical systems with infinite deformations such as flexible manipulator robots that are difficult to control due to the size of their state-space representation.

References

[1] F. Y. Wang, and Y. Gao, “Advanced Studies of Flexible Robotic Manipulators: Modeling, Design, Control and Applications,” *Series in Intelligent Control and Intelligent Automation, World Scientific. Singapore*, Vol. 4, 2003.

[2] W. J. Book, and M. Majette, “Controller design for flexible distributed parameter mechanical arms via combined state-space and frequency domain techniques,” *Transactions of ASME: Journal of Dynamic Systems, Measurement and Control*, Vol. 105, No. 4, pp. 245-254, 1983.

[3] M. O. Tokhi, and A. K. M. Azad, “Real-time finite difference simulation of a single-link flexible manipulator system incorporating hub inertia and payload,” *Proceedings of the Institution of Mechanical Engineers, Part I: Journal of Systems and Control Engineering*, Vol. 209, No. 1, pp. 21-33, 1995.

[4] J. Knani, “Dynamic modelling of flexible robotic mechanisms and adaptive robust control of trajectory computer simulation – Part I,” *Appl. Math. Model,* Vol. 26, pp. 1113-1124, 2002.

[5] M. Dado, and A. H. Soni, “A generalized approach for forward and inverse dynamics of elastic manipulators,” *Proceedings of IEEE Conference on Robotics and Automation. San Francisco,* California USA, pp. 359-364, 7–10 April 1986.

[6] S. Zribi, H. Tlijani, J. Knani, and V. Puig, “Impact of external disturbance and discontinuous input on the redundant manipulator robot behaviour using the linear parameter varying modelling approach,” *International Journal of Advanced Computer Science and Applications (IJACSA),* Vol. 8, No. 10, pp. 310-317, 2017.

[7] P. B. Usoro, R. Nadira, and S. Mahil “A Finite Element/Lagrange Approach to Modeling Lightweight Flexible Manipulators,” *ASME. J. Dyn. Sys, Meas, Control*, Vol. 108, No. 3, pp. 198-205, 1986.

[8] M. O. Tokhi, Z. Mohamed, and A. W. I. Hashim, “Modelling of a Flexible Robot Manipulator Using Finite Element Methods: A Symbolic Approach,” *Journal of Low Frequency Noise, Vibration and Active Control,* Vol. 18, No. 2, pp. 63-76,1998.

[9] T. Hsiao, and P. -H. Huang, “Iterative Learning Control for Trajectory Tracking of Robot Manipulators,” *International Journal of Automation and Smart Technology,* Vol. 7, No. 3, pp. 133-139, 17 Oct 2017.

[10] D J. D. Ratcliffe, J. J. Hätönen, P. L. Lewin, E. Rogers, and D. H. Owens. James, “Robustness analysis of an adjoint optimal iterative learning controller with experimental verification,” *International Journal of Robust and Nonlinear Control,* Vol. 18, No. 10, 2008.

[11] D. H. Owens, J. J. Hatonen, and S. Daley, “Robust monotone gradient-based discrete-time iterative learning control,” *International Journal of Robust and Nonlinear Control,* Vol. 19, No. 6, 2009.

[12] O. Houssem, B. A. Selma, and S. Salhi, “Robust Monotonic Stabilizability for Discrete Time Switched System Using D Type-Switching _Iterative Learning Control,” *The Mediterranean Journal of Measurement and Control,* Vol. 12, No. 3, pp 598-605, 2016.

[13] D. Meng, Y. Jia, J. Du, and F. Yu, “Monotonically convergent ILC systems designed using bounded real lemma,” *Taylor & Francis,* Vol. 43, No. 11, pp. 2062-2071, 2012.

[14] S. Boyd, L. El Ghaoui, E. Feron *and V. Balakrishnan*, “Linear matrix inequalities in systems and control theory,” *Society for Industrial and Applied Mathematics (SIAM),* Vol. 15, Studies in Applied Mathematics, 1994.

[15] M. Mejerbi, and J. Knani, “Influence of the motion profile on the performance of a flexible arm,” IEEE *International Conference on Control, Automation and Diagnosis (ICCAD),* pp. 076-081, Hammamet, Tunisia, 2017.