Exploiting Chaos for Fun and Profit

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***Abstract*—Chaos in dynamical systems is still considered to be a somewhat curious, and generally undesirable property of non-linear systems. Despite the plethora of chaotic control methods published over the last decades, only in a few instances has the control of chaos been used to address real world problems in engineering or medicine. This is partly due to the limits of the used control methods, which either require specific analytical knowledge of the system, or the system needs to have specific characteristics to be able to be controllable. The lack of solutions for engineering and biomedical problems may also be due to specific requirements that prevent the implementation of control methods and the, as yet unproven, benefits that controlled chaos may bring to these problems. The aim of a practical application of chaos control is to fully control chaos in theoretical problems first, and then show applicable solutions to physical problems of stability and control. This controlled chaotic state should then have clear and distinct dynamic advantages over uncontrolled chaos and steady state systems. The application of the Rate Control of Chaos (RCC) method, which is derived from metabolic control processes, has already been shown to be effective in controlling several engineering problems. RCC allows non-linear systems to be stabilised into controlled oscillations, even across bifurcations, and it also allows the system to operate in regions of the parameter space that are inaccessible without this method of control. For fun, I will show that RCC controls the N-Body problem; for profit, it can control a bioreactor model to greatly improve yield. The RCC method promises to, finally, permit the control of complex dynamic systems.**

Keywords*—*Chaos; control of chaos; bio-inspired computing

I. Introduction

The mathematical theory of non-linear systems has been developed for over a century. Due to the advances in the understanding of non-linearity, it has been realised that chaos, rather than an exception, is ubiquitous in physical systems. From low-level quantum dynamics, through molecular behaviour, and chemical oscillations, to macroscopic physical interactions, and even in astrophysics; chaos can be found virtually everywhere. Chaos exists due to the non-linear dynamics that these systems exhibit, and even though there has been a significant popular interest in the subject, the phenomenon has not yet been found to be useful, apart from creating pretty figures. The potential to exploit the dynamics of chaotic systems in chaotically controlled systems has been recognised but not yet achieved. In part, this is due to the perceived nature of chaos, which does not appear to be much different from noise in many cases. However, many forms of chaos are deterministic in the sense that their basin of attraction is limited in phase space. The chaotic system will evolve through the attractor and it will visit, eventually, all of the points of the strange attractor, although it will never repeat itself. As such, this property is not very useful, but it implies that the chaotic system will visit parts of the attractor repeatedly and very closely to previously visited points. This is exploited in many methods of chaotic control, where the controlled system is forced to revisit points in the phase space. The resulting orbit in phase space is seemingly stable and is known as an unstable periodic orbit (UPO).

Different methods of chaotic control have been proposed, that are variations of the OGY [1]–[3] method of control, or variants of analytical solutions to eliminate or reduce positive Lyapunov exponents [4], or variants of the delay feedback method of control [5]–[7]. The OGY control methods require knowledge of the unstable periodic orbits (UPOs) contained in the attractor. Therefore, an analytical understanding of the chaotic system is necessary to control the system. The delay feedback control (DFC) method uses the control function $F(y) = K(y(t) - y(t - \tau))$ which does not require any knowledge of the UPOs, but it needs appropriate choices for the control constant K and the delay τ. If K and τ are not correctly chosen then the system will not stabilise into an orbit. Some chaotic systems cannot be stabilised at all using either the single, or multiple generalised delay control method [8], [9]. Further variations of the DFC scheme, such as extended delayed feedback control [7], [10] and unstable delayed feedback control [7] may overcome this limitation by introducing additional real characteristic multipliers artificially, which greatly complicates the applicability of these control methods. It is also possible to adjust the entire system such that chaos completely disappears [4], however, that is not feasible in most physical systems and results in a not truly chaotic control, which should make only minimal changes to the chaotic system to effect control.

Exploiting chaotic dynamics for various applications has been discussed for some time. This usually concerns the use of chaos as an encoding mechanism, e.g. in encryption or radar [11]. A well known example of chaos control was the use of the OGY method to reduce cardiac fibrillation [12]. More conventional uses of chaos are rare, which is somewhat surprising given its potential. On the other hand, the route to exploitation has been unclear. Manufacturers and commercial entities tend to rely on well-known properties of the dynamic system they employ, and seek optimisation solutions in process optimisation rather than the underlying dynamics. Parameter spaces with instabilities (such as chaotic regions) are avoided, and control systems are designed to maintain the process within steady-state conditions. This makes perfect sense in the absence of suitable alternative dynamic regimes. It should, however, be considered that controlled chaos may permit wider exploration of the phase space, and with a suitable control mechanism such as the Rate Control of Chaos, this becomes possible.

A Rate Controlled Chaotic system is dynamically indistinguishable from a normal stable system, in terms of perturbation dynamics and periodic properties. However, it is not analytically dissimilar because the non-linear chaotic system is not greatly modified. The addition of the rate control term to some of the non-linear terms does not modify the global behaviour but changes the local dynamics in chaotic regions of the phase space for that variable. Clearly, this relative scaling of the coefficients changes the dynamics in part of the attractor. This can be visualised as a spatial adjustment of the stabilised system from the chaotic dynamics. It can be within the basin of attraction of the original chaotic system or the controlled system can stretch (or shrink) the attracting space where control becomes effective. The resulting controlled system has several advantages over the uncontrolled system. Firstly, the controlled system appears to be oscillating or in steady-state (for specific controlled conditions). Secondly, the control can be effective across system bifurcations, maintaining stability throughout. Thirdly, it can be enabled and disabled without further changes to the system. Fourthly, it can remain enabled even if the system is already stable, although this may cause the system to oscillate in a different orbit than without the control. Therefore, the non-linear control method can be used to control both stable and unstable non-linear systems effectively. Lastly, the presence of the control in the system allows the exploration of the parameter space such that the system can be controlled even in parameter spaces that are not accessible otherwise. For background about the Rate Control of Chaos method, see [13].

Employing the RCC method would result in a control mechanism that can be stabilised in a wide range of system parameters, the non-linear method will respond quickly, and reliably to the rapid changes due to chaotic perturbations. This has already been shown in simulations of chaotic systems, even in complex spatiotemporal chaos. Furthermore, it has direct applications to engineering problems. To demonstrate the applicability of RCC, I will first show how it can control a complex chaotic theoretical problem, the N-Body problem. This has never previously been shown to be feasible, although the practical application is not really realistic. Therefore, I will also show a practical application by the control of a bioreactor system, which demonstrates the ability of RCC to improve yield by operating in a controlled chaotic parameter domain.

II. Theoretical Application: N-Body Problem

The history of the N-Body problem is long and interesting, and harks back to the beginning of the development of mathematics to describe the motion of celestial bodies [14]. The motion of the original three bodies problem, the Earth, the Moon and the Sun, have eluded prediction for centuries [15], and only in the late twentieth century was it shown that the motion is chaotic. In fact, the entire solar system is unstable chaotic [16], [17]. Extensions of this principle in computational models to multiple (N) bodies has shown the complexity of this problem, and the difficulties in simulation and observation due to the chaotic nature of the N-Body dynamics [18], [19]. This problem is therefore interesting to apply the Rate Control of Chaos method, even though a practical implementation is not envisioned at this stage.

Using standard notation, the equation of motion for $N = n+1$ point masses, including a RCC function applied to the norm, are:

$$\sigma(\mathbf{x}_i) = e^{\left(\xi \frac{|\mathbf{x}_i - \mathbf{x}_j|}{|\mathbf{x}_i - \mathbf{x}_j| + \mu_x}\right)} \tag{1}$$

$$\ddot{\mathbf{x}}_i = -G \sum_{j=1, j\neq i}^{n} m_j \frac{\mathbf{x}_i - \mathbf{x}_j}{(\sigma(\mathbf{x}_i)\,|\mathbf{x}_i - \mathbf{x}_j| + \epsilon^2)^3} \tag{2}$$

$$i \in \{1, \ldots, n\}$$

where, $\mathbf{x}$ is the 3D position of body i, m_j the mass of body j that exerts the gravitation force on body i resulting in the acceleration of $\ddot{\mathbf{x}}_i$. G is the gravitational constant, and the control parameters are $\xi = -10$ and $\mu_x = 2000$. Equation (2) is an ordinary, coupled, non-linear differential equation of second order in time, and shows chaos due to the non-linear dynamics between the motions of the N-bodies [14]. The term ϵ^2 is a softening term used to avoid singularities when the distance between any two bodies approaches zero. The inclusion of RCC (1) provides non-linear control of the system, and when the entire system is stabilised by RCC, the softening term can be eliminated.

Videos of the simulations of almost 80,000 bodies, for both the chaotic and controlled scenarios are available at the included link[1]. To be able to distinguish between controlled and uncontrolled motion, each simulation starts initially with a random distribution of the bodies with random mass on the surface of a geometrical shape. The system is then allowed to evolve in time. When the dynamics are chaotic, the bodies increasingly occupy more space and the initial shape slowly vanishes. Note that the simulation is fully deterministic and does not contain any stochastic elements apart from the randomised initial position, and mass. The evolved system is then controlled into a stable oscillation in time and space by RCC. The controlled state becomes apparent by the repetition of the intermediate shapes and the limited space these shapes occupy. This can be repeated with different initial shapes and distributions or by switching the control off and on again which allows the system to oscillate with different intermediate shapes. In the videos of RCC, the systems evolve first for an arbitrary chosen length of 2000 time-steps chaotically. Subsequently, the control is enabled and the system quickly settles into a stable oscillation, the shapes are the same over each period but are shown rotated around the x and y axes (this has been verified numerically). Some snapshots of the states of the orbits are shown in Fig. 1, where the initial condition and intermediate states are shown for two simulations, one with a initial distribution of the bodies onto a sphere, and one with the initial distribution onto a double cone.

III. Practical Application: Bioreactors

The complexity of the metabolic processes in cellular metabolism makes it difficult to perform mathematical modelling and analysis. The complexity of the models, made of many variables and parameters, as well as the inherent non-linearity of these systems, makes if difficult to describe these systems at the level needed to perform reliably. Dynamic bioreactor models contain variable amount of detail regarding the cellular system. Mechanistic descriptions of metabolism are based on kinetic models with individual enzyme-catalytic reactions,

[1] https://goo.gl/H7nmXc

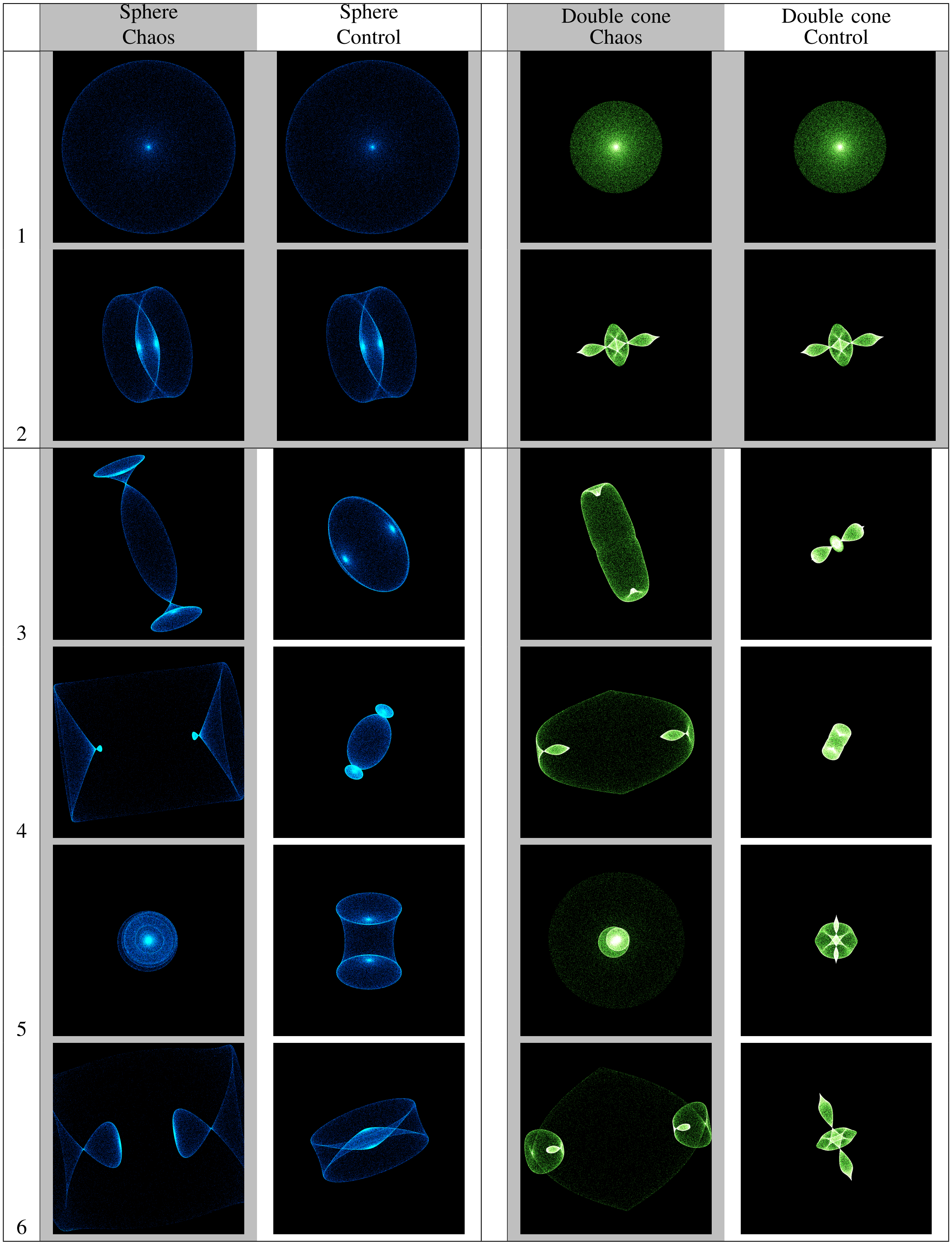

Figure 1. Chaos in the N-Body system and Rate Control of Chaos. Row 1 shows the initial distribution of bodies over a sphere or a double cone. Row 2 shows the final uncontrolled distribution of bodies before RCC is enabled for the columns 2 and 4 (white background). Rows 3-6 shows intermediate distributions of the bodies in one complete period when RCC is enabled. The images in rows 2 and 6 are the same, but rotated, for the controlled systems (columns 2 and 4). This is not the case for the uncontrolled chaos in columns 1 and 3.

which are included in dynamic mass balance equations for the system of interest. Large scale identification of enzyme kinetics, needed to accurately describe such systems, are limited by the experimental understanding of such complex dynamics. For this reason, these models are generally limited to primary metabolic pathways, and cannot encapsulate whole-cell systems. These models have not been used for bioreactor control, despite the promise of greater gain that detailed understanding of the mechanisms may bring [20].

Dynamic models that are based on limited structure descriptions of cellular metabolism, in combination with unsegregated representations of the cell populations are most suited for specific model-based controllers. These descriptions are mathematically simple, and allow the system to be described by lumped aggregation of average dynamics. These type of models have been used effectively for bioreactors for some time, however, they are still constraint by steady-state dynamics. The control objective is to maximise total production of the desired product produced in the bioreactor. Most bioreactors are equipped with sensors for online measurement of the temperature, pH level, and oxygen concentration of liquids. Simple PID regulatory loops are used to maintain the pH and temperature at constant set points predetermined to promote cell growth and product formation. The primary manipulated inputs available for higher level controllers are the nutrient flow rates, and concentrations [20].

Current methods for controlling bioreactors are based on pre-defined steady state controllers which attempt to maintain a constant environment. This is achieved by feedback control based on the measured bio-markers. Even for well known reaction systems, e.g. the production of ethanol, this is not always trivial and is sensitive to external variations that may be beyond the chosen method's ability to control. As a consequence, bioreactors are often not performing optimally in producing the wanted product or cell mass. It is also possible that the system becomes unstable and the entire batch is lost for production. Using small batch production and attempting to rigidly control all aspects of the entire process seem to be the only currently employed strategies.

Alternative methods for control are being researched with the aim to increase yield of the product or cell mass with as little sensitivity to uncontrolled variations and as little downtime for the process as possible. Suggested feasible methods are the use of neural networks [21] and the discrete time model predictive control method [22]. For the neural network control, a steady state is maintained by monitoring the bio-markers and using a predicted value generated from the network to make appropriate changes. However, this method is only effective for specific configurations and dynamic ranges and cannot be fully optimal, which is why it is not widely used. The model's predictive control method allows non-linear system to be controlled at an unstable steady state that is the desired set point for production. However, it is shown that even a well-designed control scheme may result in very poor performance [22].

In general, cells produce a metabolite on demand and in quantities only sufficient to local and current requirements. These quantities are generally small and variable which makes it difficult to extract efficiently a sufficient amount of the metabolite in an economical way. Using different methods, such as periodic forcing, high substrate concentrations, promoting factors as well as genetic modifications, the yield is attempted to increase. However, due to the complex nature of cellular regulatory systems, it is rare to find significant increases or even to maintain a stable system. The presence of higher dynamics in many of the bio-systems that are commercially interesting has been shown for some time. In particular, ethanol production [23]–[25], metabolic control [26] and nitrification [27] have shown complex and chaotic dynamics.

To demonstrate the applicability of the RCC method, a proof of concept can be designed on the basis of controlling a bioreactor experiment. It is demonstrated that a controlled simulation of a well-known chaotic metabolic reaction, based on a currently used method for controlling bioreactors, is effective, reliable and efficient. The mathematical models for simulating and controlling bioreactor systems are based on unstructured models in the form of ordinary differential equations [20]. In particular, the fermentation of sugars into alcohol by yeast, a common method to produce liquors and bioethanol, has been studied in great detail [23]. Here RCC is employed on the fermentation reactions to stabilise the chaotic dynamics into a stable oscillation at high yields. In case of periodically forced fermentors, the same method prevents the bioreactor from losing cyclic stability when it is driven by different forcing amplitudes. This property of RCC is very useful by making the bioreactor more robust and less sensitive to perturbations even at high forcing amplitudes. In Fig. 2 is shown the uncontrolled chaotic oscillation of a periodically forced fermentation. Depicted are the biomass (amount of yeast), the product concentration (ethanol), the substrate concentration (glucose) and the active component fraction which is the amount of active protein (enzymes) divided by the total biomass.

The RCC method can be applied to any or all of the variables in the differential equations that describe the fermentation process, such as the glucose influx, the ethanol efflux, the cellular generation rate and the protein production rate. To demonstrate the experimental applicability of the method, the control is only applied to the growth rate term of the substrate (13) or, alternatively, to the extraction of the product (14). The input of glucose is an externally controlled process in any case and seems the most intuitive manner to control the entire reaction system. The RCC method is effective at controlling the oscillations of the model for the entire range of parameters by appropriate changes to the rate of influx of glucose. This allows the control method to treat the entire reaction system as a black box model but is still capable of control (not included here).

An even more interesting way of controlling the fermentation process is via the regulation of the produced ethanol by controlling the rate of removal of ethanol from the bioreactor [28]. The production of ethanol by yeast is self-limiting which means that there is a limit to the concentration of ethanol that is produced by the organism as production is decreased at higher concentrations. Using this indirect means of control, which is part of the cellular biochemical control, is even more effective at controlling the dynamics of the bioreactor. The RCC method is applied to the rate of efflux of ethanol which is controlled by the manufacturer. By monitoring the amount of biomass and the fraction of active components, the rate

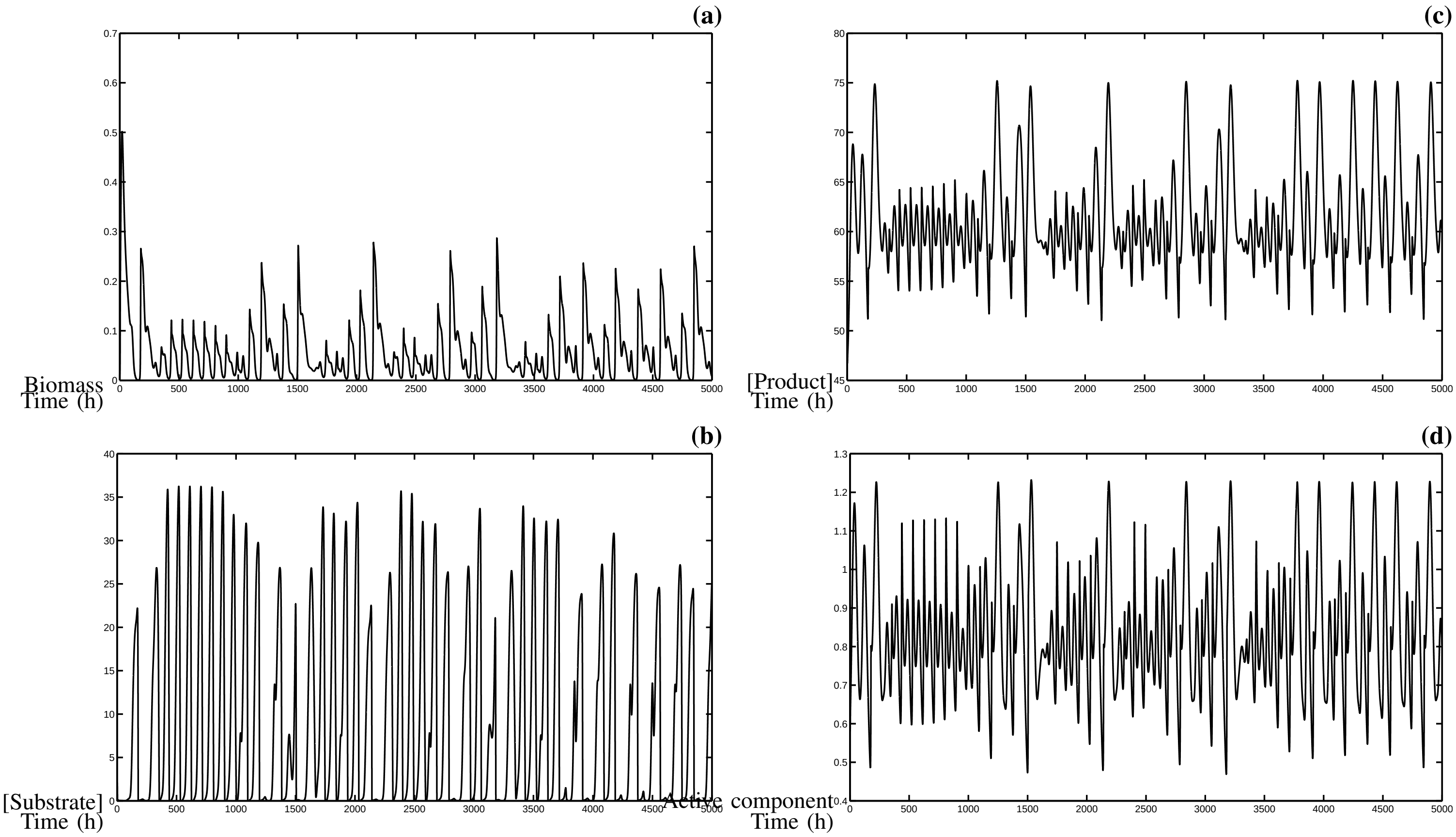

Figure 2. Chaos in a periodically forced bioreactor. (a) Biomass in time. (b) Substrate concentration (glucose) in time. (c) Product concentration (ethanol) in time. (d) Active component fraction in time.

of efflux of ethanol is appropriately reduced whenever these two components grow exponentially. Because the increase in biomass is reduced by high ethanol concentrations and the effectiveness of the enzymes is severely limited as well, a reduction in ethanol efflux will reduce the biomass increase resulting in a stable oscillation.

$$\mu = \frac{\mu_{max}\, C_s}{K_s + C_s} \tag{3}$$

$$C_{sf} = C_{s0} + A\, \sin(\omega\, t) \tag{4}$$

$$\frac{d\, C_e}{d\, t} = \frac{C_s\, C_e}{K_s + C_s}\, \left(k_1 - k_2\, C_p + k_3\, C_p^2\right) - D\, C_e \tag{5}$$

$$\frac{d\, C_p}{d\, t} = C_x\, \left(\frac{C_e\, \mu}{Y_{px}} + m_p\right) - D\, C_p\, \sigma(C_p) \tag{6}$$

$$\frac{d\, C_s}{d\, t} = \sigma(C_s)\, D\, (C_{sf} - C_s) - \left(\frac{C_e\, \mu}{Y_{sx}}\right) - m_s\, C_x \tag{7}$$

$$\frac{d\, C_x}{d\, t} = C_x\, (C_e\, \mu - D) \tag{8}$$

Equations (3) to (8) describe the fermentation process, and also include the two control functions $\sigma(C_p)$ and $\sigma(C_s)$. Only one of these functions is needed to establish control of the chaotic system (i.e. either $f_s > 0, f_p = 0$ or $f_s = 0, f_p > 0$). The forcing term in (4) is controlled by parameter A, if $A = 0$ the system reverts to the unforced system [23]. C_s is the substrate concentration, C_p the product concentration, C_e the active component fraction, C_x the biomass, μ is the specific growth rate and C_{sf} the substrate influx. The parameters of the model are $k_1 = 16, k_2 = -.497, k_3 = 0.00383, K_s = 0.5, m_s = 2.16, m_p = 1.1, \mu_{max} = 1, Y_s x = 0.024498, Y_{px} = 0.0526135, C_{so} = 140, D = 0.02$ with forcing frequency $\omega = \frac{2\pi}{93.47567}$, based on the natural period of the limit cycle. The control parameters are $\mu_1 = 100, \mu_2 = 2, \mu_3 = 1, f_s = 10, f_p = 1, \xi = -1$.

$$q_{C_s} = \frac{C_s}{C_s + \mu_1} \tag{9}$$

$$q_{C_x} = \frac{C_x}{C_x + \mu_2} \tag{10}$$

$$q_{C_e} = \frac{C_e}{C_e + \mu_3} \tag{11}$$

$$\tag{12}$$

$$\sigma(C_s) = f_s\, e^{\xi\, q_{C_s}} \tag{13}$$

$$\sigma(C_p) = f_p\, e^{\xi\, q_{C_x}\, q_{C_e}} \tag{14}$$

In Fig. 3 are shown the phase space representation of the uncontrolled chaotic attractor in (a) and the controlled oscillation in (b). This clearly shows the effectiveness of the RCC method to reduce the chaos state to a stable oscillation in the forced chaotic bio-reactor. In Fig. 4 are shown the effect

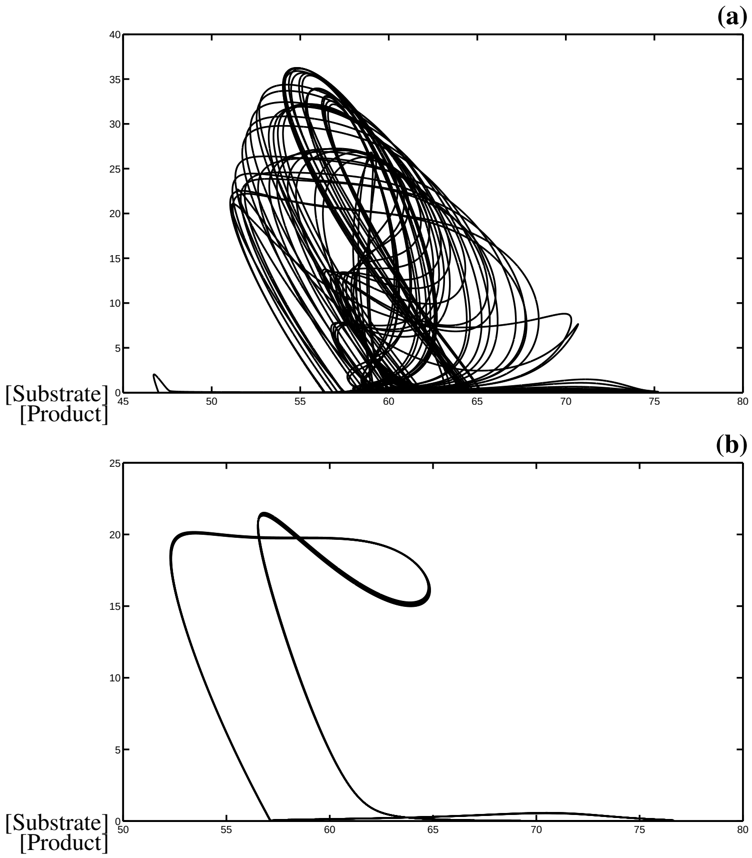

Figure 3. (a) Phase-space representations of the chaotic attractor of the bio-ethanol model, shown is concentration of product (ethanol) versus substrate (glucose). (b) Stabilised orbit within the attractor shown in (a).

of the stabilisation by the RCC method. In (a) is shown the change in substrate concentration (glucose) in time which is chaotic at first but is stabilised into a stable oscillation by the RCC method that is enabled at time 1000. In (b) in the same figure is shown the corresponding change in ethanol for the same situation.

The relation between the biomass and the RCC method is illustrated in Fig. 5 where the change in biomass in time is shown in (a) for the chaotic state but with control enabled at 1000. In Fig. 5 (b) is shown the control function $\sigma(C_p)$ (14) in time which, after an initial transient, controls the rate of ethanol efflux. Note that the change in rate is very small once the system has been stabilised into the oscillation. The change in ethanol efflux induced by the control method varies between 0 and approximately $\frac{1}{100}$ of the total efflux.

The RCC method has two additional properties that are important for this type of application. The first property is that the RCC method can be constantly enabled, but does not have any effect on the system if it is already stable. This implies that the system does not need to be constantly monitored, because the RCC method will be become involved only if the system becomes unstable and will then stabilise the system into an oscillation. The second property follows from this feature; the RCC method will maintain stability of the system even if the system is perturbed or goes through different dynamic states. This is shown in Fig. 6 where the change in product concentration (ethanol) is plotted in time. Each 1000 time steps the amplitude of the periodic forcing oscillation is increased such that the system goes through consecutive chaotic states if it was not controlled. The RCC method re-stabilises the system into different appropriate oscillations preventing the system from becoming unstable and degenerative. The dotted line indicates the average ethanol yield throughout. At the highest amplitude for the forcing oscillation (far right in the figure) the yield is significantly improved. Even at the lowest amplitude the ethanol yield is at approximately 62 g/L, which is already a significant improvement over steady state production which varies, depending on the state, but is below 50 g/L [23], [25]. Given a mean steady state production of ethanol of 46.42 g/L [23], the improved yield using the RCC method is described in Table I. Here, the steady state yield is used as 100% with the controlled forced system yield as improvements. Note that the RCC method maintains a high yield throughout the change in forcing term, even though the system would be chaotic without the RCC method for most of the forcing values in the table.

Table I. Increase in Yield Due to the RCC Method Applied to the Forced System with Average Steady State Production as Base Line 100%.

No.	Forcing term	Yield	Percentage
	N/A	46.42	100.00 %
1	30	62.43	134.48 %
2	40	65.20	140.46 %
3	50	65.47	141.04 %
4	60	65.70	141.54 %
5	70	65.03	140.09 %
6	80	70.93	152.80 %

IV. Discussion

The Rate Control of Chaos method of non-linear control allows the control of chaotic, and complex system. Its main properties are the ability to stabilise the system into a stable state, even when perturbed, or when the underlying dynamics change due to a change of parameter values. It is a feed-forward type of control that only depends on the current state of the system to control, and requires no continuous supervision. The N-Body control demonstration is illustrative of the effect the control can have on a complex interacting dynamic system, even though this specific application is only for fun, as no feasible means to apply this control to N celestial bodies is probable. The application of RCC to the bioreactor demonstrates that the control is feasible for the control of a bioreactor system, and can greatly improve the yield. This proof of concept should be relatively easily transferable to a physical demonstration. The limitations of the method are contained within the concept. It cannot target specific solutions, which makes finding efficient solutions more complicated. The control needs to be designed in combination with empirical verification to ensure reliability. So far, no chaotic or physical system that has been targeted with this control method has failed to be controlled, but it is recognised that the control is unlikely to be ubiquitous. The RCC method is not bound to specific control scenarios, as described, but can also be more widely applied to other types of control problems of complex, possibly chaotic, or chaotically perturbed systems. Further proofs of concepts are currently under investigation [29]. As a proof of the control concept, we have already developed some initial proof that RCC is capable of controlling very complex systems, such as wind turbines, or combustion

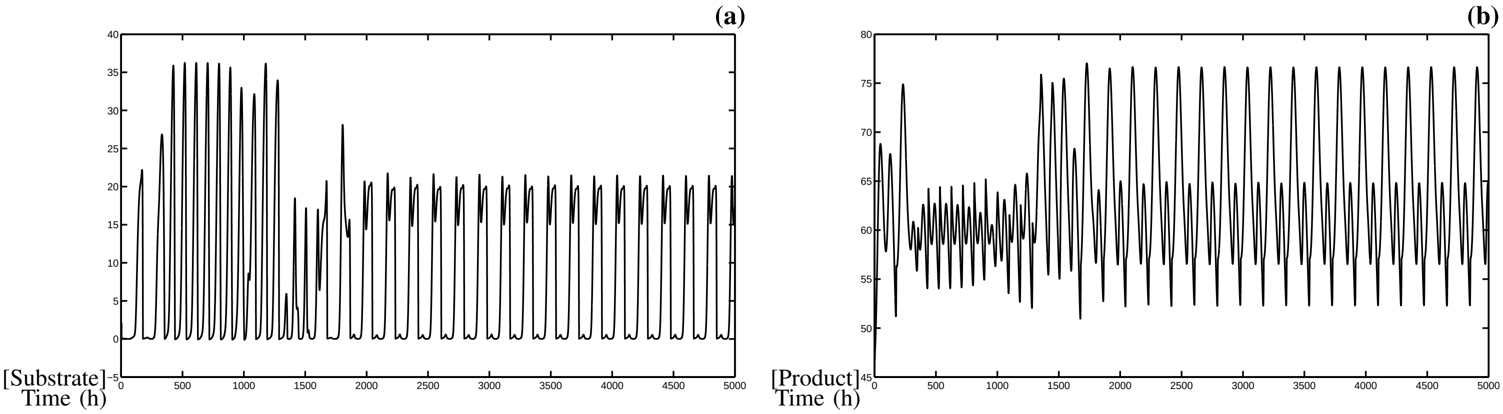

Figure 4. Stabilisation of chaos into stable oscillation, control enabled at time 1000. (a) Concentration of substrate (glucose). (b) Concentration of product (ethanol).

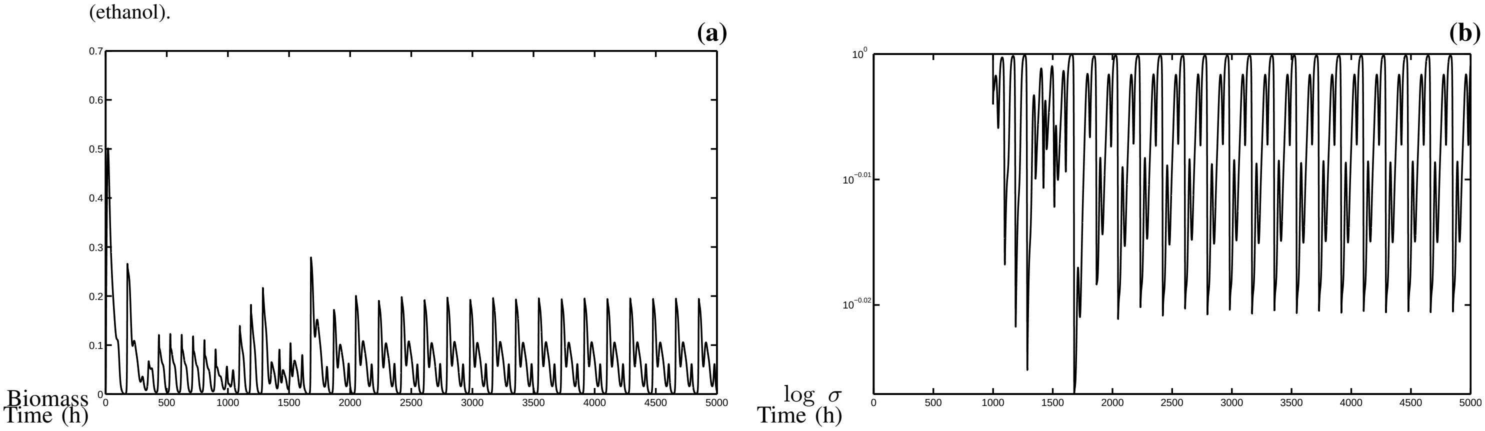

Figure 5. Stabilisation of chaos into stable oscillation, control enabled at time 1000. (a) Biomass in time. (b) Control function σ in time (vertical log scale).

engines. The method is indicated to reduce stress, and allow a system like a wind turbine to perform under a wider range of conditions than is currently feasible. The proof of concept of the combustion engine will demonstrate that it can control such a system without currently used linear control maps, which opens the possibility to allow engines to run under specific low temperature conditions, or even, to free the constraints of fuel dependency of the engine, allowing a wide range of low carbon-dioxide emission fuels to be employed in the same engine.

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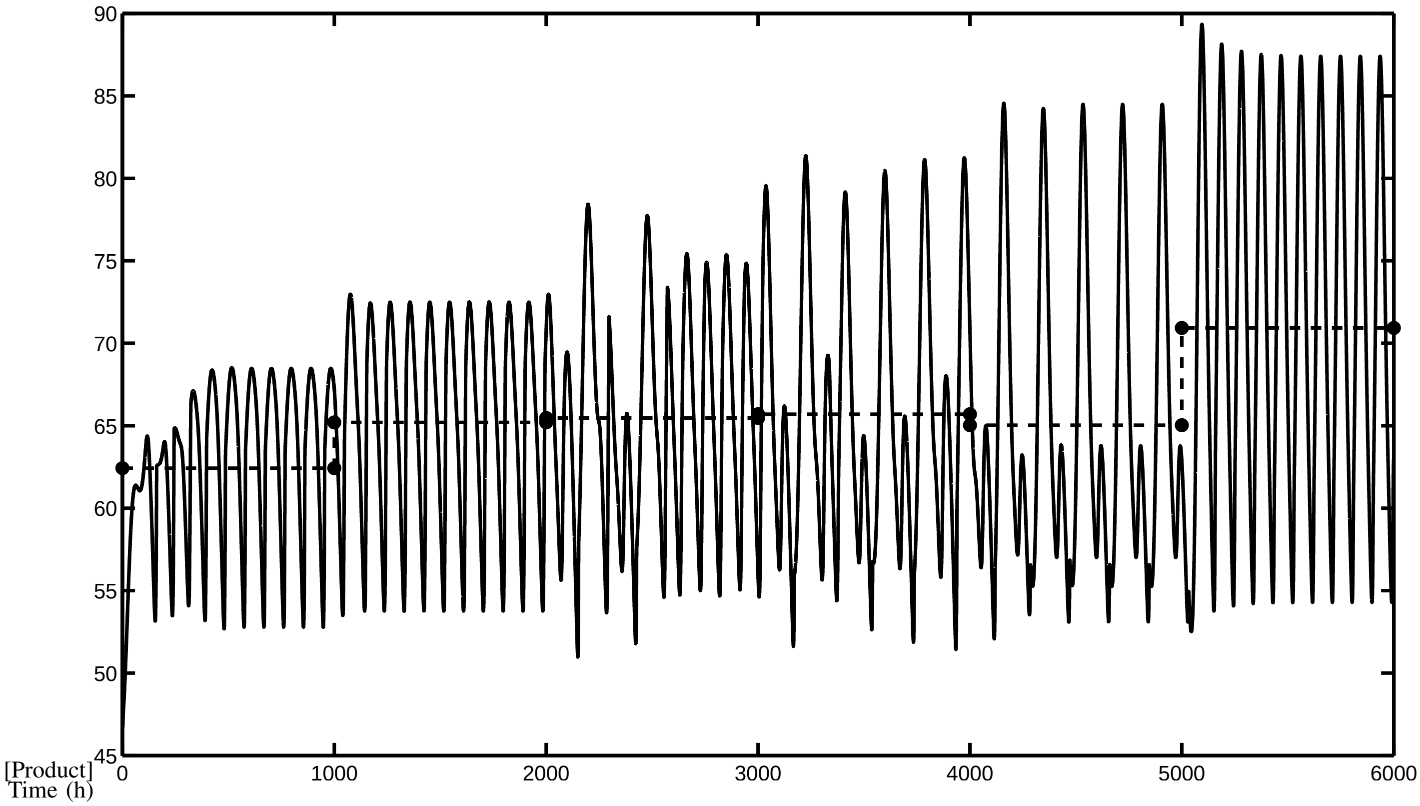

Figure 6. Stabilisation of bioreactor over several parameter changes. At every 1000h the forcing parameter is increased, making the system go through several oscillations and chaotic attractors. The control is enabled throughout and stabilises the bioreactor independent of the forcing term into stable oscillations. The dotted line indicates the yield of ethanol as mean extraction concentration at each forcing parameter change. Note that the yield may increase with higher controlled chaotic parameters (right hand side).