Quantum Computing in Geometric Algebra Formalism: Light Beam Guide Implementation

Alexander SOIGUINE
SOiGUINE Quantum Computing
Aliso Viejo, USA
E-mail: alex@soiguine.com

***Abstract*— Keeping in mind that existing problems of conventional quantum mechanics could happen due of a wrong mathematical structure, I suggest an alternative basic structure. The critical part of it is modifying commonly used terms "state", "observable", "measurement" and giving them a clear unambiguous definition. This concrete definition, along with the use of a variable complex plane, is quite natural in geometric algebra terms and helps establish a feasible language for quantum computing. The suggested approach is then used in a fiber optics quantum information transferring/processing scenario.**

Keywords—Quantum computing; geometric algebra; states; observables; measurements; light polarization

I. INTRODUCTION

B. Hiley's believed [1] that unresolved problems of conventional quantum mechanics could be the result of a wrong mathematical structure. The common wisdom of conventional quantum mechanics reads something like "The particles making up our universe are inherently uncertain creatures, able to simultaneously exist in more than one place or in more than one state". Further, "The reality of small scales creates this weirdness."

The weirdness has nothing to do with the scales. Wave-particle mysterious dualism follows from the lack of a clear distinction between operators and operands.

A lot of confusion comes from the lack of precision in using terms like "state", "observable", "measurement of observable in a state", etc. This terminology creates ambiguity because the meaning of the words differs between prevailing quantum mechanics and what is logically and naturally assumed by the human mind in scientific researches and generally used in areas of physics other than quantum mechanics. Nevertheless, I will try adhering to the terminology from the commonly accepted quantum mechanics lingo, paying respects to generations of physicists brainwashed by Bohr [4] into thinking that the question of quantum mechanics has been solved.

In classical computation scheme states are generally identified by (sets of) numbers. Every number has a binary expansion of 0's and 1's, so we can encode any input data by bit strings. Thus with a fixed length of the strings, some n , we deal with vectors in Z_2^n . Then, in its most general form [5], classical computation can be thought of as

1) The initial input $x \in Z_2^n$ encoded onto some physical system.

2) The evolution of x processed in the physical system.

3) Reading out of the computational result $f(x)$ through some measurement of the system.

In conventional quantum mechanics the steps become [6]:

1) Initialize system in some known state $|\psi_0\rangle$.

2) Unitary evolve the system until it is in some final state $U(t)|\psi_0\rangle$.

3) Measure the state of the system at the end of evolution.

In the above scheme, instead of the bit $\{0,1\} \in Z_2$ we have qubit – a quantum two-level system with two basis states $|0\rangle$ and $|1\rangle$. Qubit is formally an element of two dimensional complex Hilbert space

$$C^2 \ni \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = z_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + z_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = z_1|0\rangle + z_2|1\rangle \quad . \tag{1}$$

The state $|\psi_0\rangle$ is then an element of $\left(C^2\right)^n$ [5], if the same assumption of a fixed length of qubit strings is made.

In the suggested approach a qubit state will be lifted to g-qubit, element of G_3^+ - even geometric subalgebra of the geometric algebra G_3 in three dimensions. The lift particularly uses the generalization of a formal imaginary plane to explicitly defined planes in three dimensions [2], [3]. The g-qubit states are interpreted strictly as operators acting on observables, also elements of geometric algebra, in the process of measurement. That follows Dirac's seminal idea [7] to remove the distinction between an element of the operator algebra and the wave function (state) without losing any information about the content of what is carried by the wave function.

Thus, the suggested computational scheme becomes:

1) Initialize system in some known state $\left(G_3^+\right)^n$ which is a set of operators acting on observables composed from elements of G_3 .

2) Evolve the system until it is in some final state.

3) Identify the state of the system at the end of evolution by acting with the operators comprising the final state on observables.

Since the degrees of freedom of just one g-qubit give infinite number of available values, implementation of the simplest case $n = 1$ would be of great importance.

In the case of electromagnetic field its state, considered as element of geometric algebra, acts (operates) on other physical entities which can also be electromagnetic fields.

II. QUBIT STATES IN GEOMETRIC ALGEBRA

The Dirac's idea is exactly and accurately implemented in the case of a g-qubit when the action of a state on observable is non-commutative operation:

$$(S(\lambda), O(\mu)) \to O(\nu) \xleftarrow{def} O(\nu) = S^{-1}(\lambda) O(\mu) S(\lambda), \tag{2}$$

where $S(\lambda)$ are elements of even sub-algebra G_3^+ of geometric (Clifford) algebra G_3 over three dimensional Euclidean space [3] and $O(\mu)$, $O(\nu)$ are generally elements of G_3.

The even sub-algebra G_3^+, in the fiber bundle terms, can be taken as total space for base space C^2 and any C^2 qubit $\begin{pmatrix} x_1 + iy_1 \\ x_2 + iy_2 \end{pmatrix}$ has fiber in G_3^+. The construction is as follows:

Let (B_1, B_2, B_3) be an arbitrary triple of unit value mutually orthogonal bivectors in three dimensions satisfying, in the assumption of a right-hand screw orientation, the identity $B_1 B_2 B_3 = 1$ [1] and multiplication rules:

$$B_1 B_2 = -B_3, B_1 B_3 = B_2, B_2 B_3 = -B_1 \tag{3}$$

The elements of the fiber are g-qubits[2] defined by the map:

$$\begin{pmatrix} x_1 + iy_1 \\ x_2 + iy_2 \end{pmatrix} \Rightarrow x_1 + y_1 B_1 + y_2 B_2 + x_2 B_3 =$$

$$x_1 + y_1 B_1 + y_2 B_1 B_3 + x_2 B_3 = x_1 + y_1 B_1 + (x_2 + y_2 B_1) B_3 \tag{4}$$

The fiber reference frame (B_1, B_2, B_3) can be arbitrary rotated in three dimensions. In that sense we have principal fiber bundle $G_3^+ \to C^2$ with the standard fiber as a group of rotations which is also effectively identified by elements of G_3^+.

Fiber element

$$x_1 + y_1 B_1 + y_2 B_2 + x_2 B_3 = x_1 + y_1 B_1 + (x_2 + y_2 B_1) B_3 \tag{5}$$

is the geometric algebra sum of two items, $x_1 + y_1 B_1$ and $(x_2 + y_2 B_1) B_3$: the first is the fiber element corresponding to conventional quantum mechanical state $|0\rangle$, in usual Dirac notations, and the second one – corresponding to $|1\rangle$.

State $x_1 + y_1 B_1$ when acting on a G_3^+ observable does not change the B_1 component of an observable and only rotates other two components of the bivector part belonging to the subspace spanned by B_2 and B_3 [3], [8].

State $(x_2 + y_2 B_1) B_3$ structurally differs from $x_1 + y_1 B_1$ by factor B_3, which makes flip of the result of the action of $x_2 + y_2 B_1$ on observable over the plane B_1, changing the sign of the B_1 component.

Thus the actual geometrical sense of the G_3^+ fiber states corresponding to conventional quantum mechanical basis states $|0\rangle$ and $|1\rangle$ is that the first one only rotates observable around an axis orthogonal to some arbitrary given plane in three dimensions, while the second one additionally flips the result, after rotation, over that plane.

III. EVOLUTION OF THE G-QUBIT STATES

It is plausible to retrieve how the Hamiltonian action on states in conventional quantum mechanics is generalized in the current context.

Any conventional quantum mechanics (CQM) C^2 state lift to G_3^+ can be written as exponent:

$$\begin{pmatrix} x_1 + iy_1 \\ x_2 + iy_2 \end{pmatrix} \Rightarrow$$

$$x_1 + y_1 B_1 + y_2 B_2 + x_2 B_3 =$$

$$x_1 + \sqrt{1 - x_1^2} \left(\frac{y_1}{\sqrt{1 - x_1^2}} B_1 + \frac{y_2}{\sqrt{1 - x_1^2}} B_2 + \frac{x_2}{\sqrt{1 - x_1^2}} B_3 \right) =$$

$$\cos\varphi + (b_1 B_1 + b_2 B_2 + b_3 B_3) \sin\varphi = e^{I_S \varphi} \tag{6}$$

where $\cos\varphi = x_1$, $\sin\varphi = \sqrt{1 - x_1^2}$, $b_1 = \frac{y_1}{\sqrt{1 - x_1^2}}$, $b_2 = \frac{y_2}{\sqrt{1 - x_1^2}}$, $b_3 = \frac{x_2}{\sqrt{1 - x_1^2}}$, $I_S = b_1 B_1 + b_2 B_2 + b_3 B_3$. (7)

Hamiltonian in CQM is a self-adjoint matrix of general form:

$$H = \begin{pmatrix} \alpha + \beta_1 & \beta_2 - i\beta_3 \\ \beta_2 + i\beta_3 & \alpha - \beta_1 \end{pmatrix} \tag{8}$$

It acts on two-dimensional complex vectors according to the usual rules of linear algebra, $|\psi\rangle \to H|\psi\rangle$.

---

[1] The reference frame (B_1, B_2, B_3) can be chosen as left-hand screw oriented, $B_1 B_2 B_3 = -1$. It is just a reference frame and has nothing to do with the physical nature of three dimensional space.

[2] The element of fiber depends on which basis bivector is chosen to define "complex plane". Cyclic permutation of the reference frame bivectors delivers different elements.

The Hamiltonian matrix geometric algebra lift[3] is not an element of G_3^+, since it has the form $\alpha + I_3(\beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3)$. We can forget about α as it may only affect final multiplication by a scalar. Then

$$H_G \overset{def}{\equiv} I_3(\beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3) = I_3\sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2} \cdot \left(\frac{\beta_1 B_1}{\sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2}} + \frac{\beta_2 B_2}{\sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2}} + \frac{\beta_3 B_3}{\sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2}} \right) = I_3|H_G| I_{H_G} = I_3|H_G| e^{\frac{\pi}{2} I_{H_G}} \tag{9}$$

Thus we see that multiplication of a complex two-dimensional vector by matrix H corresponds, if mapped directly to multiplication in G_3, to the operation:

$$I_3|H_G| e^{\frac{\pi}{2} I_{H_G}} e^{I_S \varphi} \tag{10}$$

It does not look good since the result does not belong to G_3^+, our space of states. This means that the action of Hamiltonians, as matrices, on the C^2 states using linear algebra multiplication, cannot be equivalent to multiplication of results of lifts in G_3.

Due to that we need to look for other options of lifting the operation $H\begin{pmatrix} x_1 + iy_1 \\ x_2 + iy_2 \end{pmatrix}$ to G_3^+. Actually, we have the two:

1) Rotation of a G_3^+ element, particularly a state, in the plane of the Hamiltonian lift by the angle defined by the Hamiltonian value.

2) Clifford translation of a G_3^+ element, particularly a state, along the big circle of the S^3 sphere. The circle is an intersection of the sphere with the plane of the Hamiltonian lift.

Let's initially consider the second option.

Instead of unitary transformations acting on the Hilbert space vector states of C^2 transforming them into new states, $U(t)|\psi_0\rangle$, the corresponding transformations acting in the fiber bundle with total space G_3^+ over base space C^2 are given, if the Hamiltonian depends on time, as sequences of infinitesimal Clifford translations [3][4]:

$$|s(t+\Delta t)\rangle_{(\alpha,\beta,I_S)} = e^{-I_3 H(t)\Delta t}|s(t)\rangle_{(\alpha,\beta,I_S)} = e^{-\left(I_3 \frac{H(t)}{|H(t)|}\right)|H(t)|\Delta t}|s(t)\rangle_{(\alpha,\beta,I_S)} \tag{11}$$

where I_3 is unit value oriented volume in the three dimensions and $H(t)$ - the Hamiltonian expanded in basis (I_3B_1, I_3B_2, I_3B_3). Unit value bivector $I_3\frac{H(t)}{|H(t)|}$ is a generalization of an imaginary unit explicitly defining the plane of the S^3 sphere big circle.

Remark 3.1: If the Hamiltonian does not depend on time a finite Clifford translation gives:

$$|s(t)\rangle_{(\alpha,\beta,I_S)} = e^{-I_3 H t}|s(0)\rangle_{(\alpha,\beta,I_S)} = e^{-\left(I_3 \frac{H}{|H|}\right)|H|t}|s(0)\rangle_{(\alpha,\beta,I_S)} \tag{12}$$

The geometric algebra framework with an arbitrary variable plane of state bivector (VPSB) generalizes geometrically unspecified complex plane of CQM. Thus, it follows that the CQM Schrodinger equation $\hat{H}|\psi(t)\rangle = i\frac{\partial}{\partial t}|\psi(t)\rangle$ in the VPSB framework takes the form:

$$I_3 H(t)|s(t)\rangle_{(\alpha,\beta,I_S)} = -\frac{\partial}{\partial t}|s(t)\rangle_{(\alpha,\beta,I_S)} \tag{13}$$

with generally varying bivector $I_3H(t)$. It follows that the arbitrary state transformation is the holonomy $\int_L e^{I_{H(l)}|H(l)|dl}|s\rangle_{(\alpha,\beta,I_S)}$ where the integral is taken along the Hamiltonian vector curve trace on the surface of unit sphere S^3 [3].

The critical thing to remember: Schrodinger equation in geometric algebra terms is an equation defining evolution of states, operators. The states act on other states either via the Clifford translations producing new states, or on G_3^+ elements, interpreted as observables, when executing measurements, which are rotations.

In Section V, will be discussed the options for rotation in the plane of the Hamiltonian G_3^+ lift by the angle defined by Hamiltonian value.

IV. Electric Field Polarization

To deal with the guided light beams as physical processes carrying information about the states in the geometric algebra sense I start with the electromagnetic fields and their polarizations in the G_3^+ terms [9], [10].

What is different in the current approach to the light propagation in a beam guide is the fact that formally used imaginary unit is replaced with a unit bivector in a three

[3] See the Hamiltonian lift calculation in [3].

[4] I will use H instead of H_G since only the geometric algebra meaning of a Hamiltonian will be used .

dimensional space not necessary orthogonal to the z direction, default beam guide axis.[5] The electric fields should naturally be considered as states, up to the magnitude factor, that's the G_3^+ operators acting on other states or on observables.

Assume we deal with a detectable polarization in the xy plane: circular, elliptic or linear one, which means that the electric field vector end point moves along the corresponding trajectory. The following result emerges:

Theorem 4.1. Any type of polarization in the xy plane is a projection of circular polarization in some plane S.

Proof:

Since the plane of rotation/oscillation of electric field vector may be any plane in three dimensions, the plane of polarization should be explicitly defined[6].

Electric field can evolve in a plane of some unit bivector I_S being in the state of circular polarization in that plane.

Suppose polarization is measured in xy plane and is an ellipse of a general parametric form:

$$(a\cos\alpha\cos t - b\sin\alpha\sin t)\hat{x} + (a\sin\alpha\cos t + b\cos\alpha\sin t)\hat{y}$$

$$0 \le t \le 2\pi, \tag{14}$$

where α is an angle of the ellipse $a\cos t\hat{x} + b\sin t\hat{y}$, $0 \le t \le 2\pi$, rotation in xy plane relative to the x direction; a is the value of the ellipse semi axis along the direction of the x axis; b is semi axis along the orthogonal direction, $\hat{x}$ and $\hat{y}$ are unit vectors along corresponding axes.

Remark 4.1: In pure geometric algebra terms the rotation of ellipse with semi axes parallel to coordinate axes by angle α is $(a\cos t\hat{x} + b\sin t\hat{y})e^{\hat{x}\hat{y}\alpha}$ (multiplication from the right!).

Remark 4.2: If $a = b$ (circle) the rotation gives the same circle. If one of semi axes, say b, is zero, we get vector oscillating with the amplitude a along the line $(\cos\alpha)\hat{x} + (\sin\alpha)\hat{y}$ (or with the amplitude b along the line $-(\sin\alpha)\hat{x} + (\cos\alpha)\hat{y}$ if $a = 0$). This is the case of a linear polarization.

Assume the normal to I_S is received from the normal to xy plane (z direction) by rotation by angle θ in a plane S_R passing through the major semi axis of the ellipse (14). Define angle of rotation by $\cos(\theta) = \dfrac{\min(a,b)}{\max(a,b)}$. The plane of rotation is defined by a unit bivector dual to unit vector along minor semi axis. If major semi axes has value a then

$$I_{S_R} = -(\sin\alpha)I_3\hat{x} + (\cos\alpha)I_3\hat{y} \tag{15}$$

If major semi axes has value b,

$$I_{S_R} = (\cos\alpha)I_3\hat{x} + (\sin\alpha)I_3\hat{y} \tag{16}$$

Thus, the two unit bivectors for the xy plane and S plane are received from each other as:

$$I_S = e^{-I_{S_R}\frac{\theta}{2}} I_{xy} e^{I_{S_R}\frac{\theta}{2}}, \quad I_{xy} = e^{I_{S_R}\frac{\theta}{2}} I_S e^{-I_{S_R}\frac{\theta}{2}} \tag{17}$$

The projection of the I_S polarization circle, expanded to radius $\max(a,b)$, onto xy plane is exactly the original ellipse (14). *QED.*

The circular polarized electromagnetic wave states actually comprise the basis for all other types of polarizations because they are the only type of waves that come from the solution of Maxwell equations in free space when done accurately in geometric algebra terms.

Let's take the electromagnetic field in the form

$$F = F_0 \exp\left[I_S\left(\omega t - \vec{k}\cdot\vec{r}\right)\right] \tag{18}$$

with the only requirement being that it satisfies the Maxwell system of equations in free space, which in geometrical algebra terms takes the form of one equation:

$$(\partial_t + \nabla)F = 0 \tag{19}$$

using geometric algebra multiplication.

Element F_0 in (18) is a constant element of geometric algebra G_3, undefined yet, and I_S is a unit value bivector of plane S in three dimensions, a generalization of an imaginary unit in the current approach.

Electromagnetic field can be identified by geometric algebra sum of a vector E, the electric field, and bivector $I_3 B$, the magnetic field. That means that to retrieve the structure of the element F_0 we need to compare the right hand side of (18) with the geometric algebra element $E + I_3 B$.

The exponent in (15) is a unit value element of G_3^+ with the I_S bivector plane: $e^{I_S\varphi} = \cos(\varphi) + I_S\sin(\varphi)$, $\varphi = \omega t - \vec{k}\cdot\vec{r}$. Since no assumptions about F_0 exist, we will generally write: $F_0 = (F_0)_S + (F_0)_V + (F_0)_B + (F_0)_P$.

The geometric algebra product

$$F_0 e^{I_S\varphi} = \left[(F_0)_S + (F_0)_V + (F_0)_B + (F_0)_P\right]\left[\cos(\varphi) + I_S\sin(\varphi)\right] \tag{20}$$

should give $E + I_3 B$, which is the sum of a vector and bivector.

[5] Interest to the transverse light beam spin models is growing intensively, see, for example [14].

[6] Similar definitions of polarization are used in different contexts, see, for example [13].

First, it follows that $(F_0)_S$ and $(F_0)_P$ must be zeros.

Second, the product $(F_0)_V I_S$ is sum of a vector and pseudoscalar (see [3], Sec.1.3).The pseudoscalar must be zero, which is only possible when $(F_0)_V$ lies in the plane of I_S. The remaining vector part of the product is equal to the cross product of $(F_0)_V$ and vector dual to I_S, that's $(F_0)_V \wedge I_3 I_S$ which is the vector $(F_0)_V$ rotated by $\frac{\pi}{2}$ in the positive direction in plane S.

Finally, the product of two bivectors $(F_0)_B I_S$ is the sum of the scalar, equal to the scalar product of two vectors, which are dual correspondingly to $(F_0)_B$ and I_S, and the bivector dual to vector $I_3(F_0)_B \wedge I_3 I_S$. The scalar part must be zero which means that the bivector planes are orthogonal. Then the remaining bivector $I_3\left(I_3(F_0)_B \wedge I_3 I_S\right)$ is the $(F_0)_B$ rotated by $\frac{\pi}{2}$ around the axis orthogonal to the plane S.

Thus, the geometric algebra element F is geometric algebra sum of a vector in plane S and bivector orthogonal to that plane. Both rotate synchronically with the angle $\varphi = \omega t - \vec{k} \cdot \vec{r}$ around the axis orthogonal to plane S and lying in the plane of the bivector.

This rotation defines circular polarization in plane S, thus justifying the practical applicability of the earlier results that any polarization in the xy plane can be received as a projection of circular polarization in some plane.

V. HAMILTONIAN ACTION AS ROTATION

An electric field defined by a vector rotating in plane S is obviously a state (up to a real constant multiplier, amplitude) in the G_3^+ terms.

Below we consider the situation that is usually defined as spin and orbital angular momenta, or alternatively, chirality.

An arbitrary spin angular momentum is defined by the result of the inclination of the electric field vector rotating in the xy plane. The orbital angular momentum appears when the inclined plane rotates around the z axis. Thus we have composition of inclination of unit bivector, $I_S = e^{-I_{S_R}\frac{\theta}{2}} I_{xy} e^{I_{S_R}\frac{\theta}{2}}$, and further rotation.

The plane of the initial inclination $I_{S_R} = -(\sin\alpha) I_3 \hat{x} + (\cos\alpha) I_3 \hat{y}$ can be taken with $\alpha = 0$ because the projection polarization ellipse rotates, permanently by default, in the xy plane. So we assume $I_{S_R}\big|_{\alpha=0} = -(\sin\alpha) I_3 \hat{x} + (\cos\alpha) I_3 \hat{y}\big|_{\alpha=0} = I_3 \hat{y} = I_{zx}$. With the typical identification of basis bivectors: $B_1 = \hat{y}\hat{z}$, $B_2 = \hat{z}\hat{x}$, $B_3 = \hat{x}\hat{y}$, we get[7] $I_S = e^{-B_2\frac{\theta}{2}} B_3 e^{B_2\frac{\theta}{2}}$:

The rotation of the inclined polarization state around the z axis produces the result:

$$e^{-B_3\frac{\varphi(t)}{2}} e^{-B_2\frac{\theta}{2}} B_3 e^{B_2\frac{\theta}{2}} e^{B_3\frac{\varphi(t)}{2}} \tag{21}$$

where $\varphi(t)$ varies with time angle of rotation around the z axis.

The inclination gives $B_1 \sin\theta + B_3 \cos\theta$. Then the subsequent result of the state rotation around the z axis has an explicit bivector form:

$$\begin{aligned} U(\theta,\varphi) &= B_1 \sin\theta\cos\varphi + B_2 \sin\theta\sin\varphi + B_3\cos\theta = \\ &B_1 e^{B_3\varphi} \sin\theta + B_3\cos\theta \end{aligned} \tag{22}$$

VI. ROTATION OF THE CIRCULAR POLARIZATION PLANE AND LINEAR POLARIZATION IN THE PROJECTION PLANE

Consider the case $\theta = \frac{\pi}{2}$ which results in linear polarization along the line rotated by φ angle relative to x axis in the xy plane. The circular polarization plane contains the z axis and is rotated around it with the second (external) rotation in the transformation, measurement of B_3 in the state $e^{B_2\frac{\pi}{4}} e^{B_3\frac{\varphi}{2}}$:

$$e^{-B_3\frac{\varphi}{2}} \left(e^{-B_2\frac{\pi}{4}} B_3 e^{B_2\frac{\pi}{4}} \right) e^{B_3\frac{\varphi}{2}} \tag{23}$$

Assume we have physical mechanism of rotating circular polarized electric field, state, in the xy plane, in other words a mechanism sufficient to executing the measurements:

$$\begin{aligned} &e^{-B_3\frac{\varphi}{2}} \left(e^{-B_2\frac{\pi}{4}} B_3 e^{B_2\frac{\pi}{4}} \right) e^{B_3\frac{\varphi}{2}} = e^{-B_3\frac{\varphi}{2}} B_1 e^{B_3\frac{\varphi}{2}} = \\ &B_1\cos\varphi + B_2\sin\varphi \end{aligned} \tag{24}$$

The result has a zero value bivector component in the plane of B_3, as it should, although geometrically the projection of the circle onto xy plane is a degenerated ellipse - a straight segment of unit length centered at 0 of the line along the vector $(\cos\varphi)\hat{x} + (\sin\varphi)\hat{y}$. Obviously, this linear polarization line rotates together with angle φ. The

[7] For convenience we ignore that the inclined bivector does not have a unit value and more accurately should be $|E| e^{-B_2\frac{\theta}{2}} B_3 e^{B_2\frac{\theta}{2}}$.

information about circular polarization sense is, by the way, lost.

Generally we have two circular polarization states, left-hand and right-hand, and the above formula for the opposite circular polarization sense is:

$$e^{-B_3\frac{\varphi}{2}}\left(e^{-B_2\frac{\pi}{4}}\left(-B_3\right)e^{B_2\frac{\pi}{4}}\right)e^{B_3\frac{\varphi}{2}} = e^{-B_3\frac{\varphi}{2}}\left(-B_1\right)e^{B_3\frac{\varphi}{2}} = -B_1\cos\varphi - B_2\sin\varphi \qquad (25)$$

Thus, the line of linear polarization in the xy plane remains the same. This fact raises a separate question as distinguishing between two circular polarizations. Both can be the origin of the same linear polarization and this is critically important for a basic algorithm of function value calculations, which will be demonstrated in next section. Thus, we are making the assumption that there is only one circularly polarized mode of the spin angular momentum, for example in the B_1 sense.

VII. Evolving States Via Transformations of Circular Polarization States

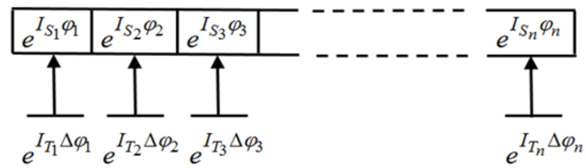
Now assume that transformation of the state B_1, spin angular momentum, is made by the Clifford translation with a Hamiltonian $H(t)$ as formulated earlier:

$$\left|s(t+\Delta t)\right\rangle_{(\alpha,\beta,I_S)} = e^{-I_3 H(t)\Delta t}\left|s(t)\right\rangle_{(\alpha,\beta,I_S)} = e^{-\left(I_3\frac{H(t)}{|H(t)|}\right)|H(t)|\Delta t}\left|s(t)\right\rangle_{(\alpha,\beta,I_S)} \qquad (26)$$

To keep up with the orbital angular momenta corresponding to the external transformation of the polarization plane the plane of bivector $I_3\dfrac{H(t)}{|H(t)|}$ is supposed to be constant and equal to B_3. Physically, such Hamiltonian action can be implemented via a magnetic field parallel to the light guide, for example as an electric current coil around the light guide.

The core of quantum computing should not be in entanglement, which in conventional quantum mechanics comes from the representation of the many particle states as tensor products of individual particle states. The core of the quantum computing scheme should be in the manipulation and transferring of quantum states as operators decomposed in geometrical algebra variant of qubits (g-qubits), or alternatively four dimensional unit sphere points. This way the quantum computer is, as it should be, an analog computer keeping information in sets of objects with infinite number of degrees of freedom, contrary to the two-value bits or two-dimensional Hilbert space elements, qubits.

In the suggested computational scheme, defined in Section 1, we write the initial state $\left(G_3^+\right)^n$ as $\left(e^{I_{S_1}\varphi_1}, e^{I_{S_2}\varphi_2}, \ldots, e^{I_{S_n}\varphi_n}\right)$. As such, similar to the traditional Turing machine scheme, one can schematically represent a $\left(G_3^+\right)^n$ state evolution as



States $e^{I_{T_k}\Delta\varphi_k}$ realizing evolution, act on the components $e^{I_{S_k}\varphi_k}$ of the initial state as Clifford translations:

$$e^{I_{S_k}\varphi_k} \rightarrow e^{I_{T_k}\Delta\varphi_k} e^{I_{S_k}\varphi_k} \qquad (27)$$

If a continuous sequence of such translations takes place we get the holonomy formulated in Section 3:

$$\int_L e^{I_{T(l)}d\varphi(l)}\left|s\right\rangle_{\left(\cos\varphi(l),\sin\varphi(l),I_{S(l)}\right)} \qquad (28)$$

If the transformation (27) is taken as an infinitesimal one (or with a stable plane T_k), the state $e^{I_{S_k}\varphi_k}$ is rotated in the plane T_k by the angle $\Delta\varphi_k$ and synchronically rotated by the same angle in two planes orthogonal to T_k in three dimensions [3].

Suppose we have a light guide with the input of series of length n time bins bearing states $e^{I_{S_k}\varphi_k}$ [8]. Time bin items are transformed according to rules (27). The output state, final state in terms of the suggested scheme, acts on n copies of the observable B_1. The result is a series of length n of linear polarizations in the xy plane.

That is all true in our simple case of the spin angular momentum orthogonal to the z axis. Other arbitrary directions will produce more sophisticated options.

The suggested computational scheme is applicable, for example, to function calculations.

The light guide single mode input, discrete in time, of the calculated function argument is identified by the time step number (index, time stamp t_k) and the time bin state item plane and angle. In the current case the latter two are B_3 and the angle of rotation of polarization B_1 around the z axis.

Clifford translations acting on the state items all have the same plane - $T_k = B_3$, and angles of rotations are defined by Hamiltonian values $\left|H(t_k)\right|$. In the output we have a sequence

[8] Similar time bins scheme, though with much simpler bin items, was considered, for example, in [11] .

of length n of the final state items $e^{I_{B_3}|H(t_k)|}e^{I_{B_3}t_k}$, $1 \le k \le n$.

The measurement phase (the last item of our computational scheme defined in Section 1) is the set of measurements:

$$e^{-I_{B_3}(t_k+|H(t_k)|)}B_1 e^{I_{B_3}(t_k+|H(t_k)|)} \tag{29}$$

The sequence of length n of linear polarizations in the xy plane as defined by (6.1):

$$B_1 \cos[2(t_k + |H(t_k)|)] + B_2 \sin[2(t_k + |H(t_k)|)],\ 1 \le k \le n \tag{30}$$ [9]

VIII. Conclusions

Two seminal ideas – variable and explicitly defined complex plane in three dimensions, and the G_3^+ states[10] as operators acting on observables – allow to put forth comprehensive and much more detailed formalism appropriate for quantum mechanics in general and particularly for quantum computing schemes. Based on this new mathematical structure, the suggested computational scheme is implemented in terms of the guided light polarization variant of geometric algebra g-qubits. The approach may be thought of, for example, as a far-reaching geometric algebra generalization of some proposals for quantum computing formulated in terms of light beam time bins, see [11], [12], but offering much more strength and flexibility in practical implementation.

References

[1] B. J. Hiley, "Structure Process, Weak Values and Local Momentum," Journal of Physics: Conference Series, vol. 701, no. 1, 2016.

[2] M. Soiguine, "Complex Conjugation - Relative to What?," in Clifford Algebras with Numeric and Symbolic Computations, Boston, Birkhauser, 1996, pp. 284-294.

[3] Soiguine, Geometric Phase in Geometric Algebra Qubit Formalism, Saarbrucken: LAMBERT Academic Publishing, 2015.

[4] M. Gell-Mann, in The Nature of Physical Universe: the 1976 Nobel Conference, New York, Wiley, 1979, p. 29.

[5] Z. Wang, Topological Quantum Computation, Providence: American Mathematical Society, 2010.

[6] Nayak, S. H. Simon, A. Stern, M. Freedman and S. Das Sarma, "Non-Abelian Anyons and Topological Quantum Computation," Rev. Mod. Phys., vol. 80, p. 1083, 2008.

[7] P. A. M. Dirac, "A new notation for quantum mechanics," Mathematical Proceedings of the Cambridge Philosophical Society, vol. 35, no. 3, pp. 416-418, 1939.

[8] Soiguine, "What quantum "state" really is?," June 2014. [Online]. Available: http://arxiv.org/abs/1406.3751.

[9] Soiguine, Vector Algebra in Applied Problems, Leningrad: Naval Academy, 1990 (in Russian).

[10] J. W. Arthur, Understanding Geometric Algebra for Electromagnetic Theory, Hoboken: John Wiley & Sons, 2011.

[11] P. C. Humphreys et al, "Linear Optical Quantum Computing in a Single Spatial Mode," Physical Review Letters, vol. 111, no. 150501, 2013.

[12] P. Kok et al, "Linear optical quantum computing with photonic qubits," Rev. Mod. Phys., vol. 89, pp. 135-174, 2007.

[13] S. Bravyi and A. Kitaev, "Universal quantum computation with ideal Clifford gates and noisy ancillas," 2004. [Online]. Available: arxiv:quant-ph/0403025v2.

[14] Aiello and P. Banzer, "The ubiquitous photonic wheel," Journal of Optics, vol. 18, no. 8, 2016.

[9] Default right hand screw circular polarization in the spin angular momentum plane is taken. The ambiguities in the values of function arguments and function values due to the fact that both have ranges in finite intervals can be removed via $t_k \to t_k(\mathrm{mod}(2\pi))$ and $|H(t_k)| \to |H(t_k)|(\mathrm{mod}(2\pi))$.

[10] Good to remember that state and wave function are actually (at least should be) synonyms in conventional quantum mechanics.