Prediction Intervals based on Doubly Type-II Censored Data from Gompertz Distribution in the Presence of Outliers

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***Abstract*—The study aims at getting the Bayesian predication intervals for some order statistics of future observations from the distribution of Gompertz (Gomp (α, β)). Doubly Type-II censored data has assisted obtaining in the presence of single outlier that arose from the different same family members of distribution. Single outlier of type $\beta\,\beta_0$ and $\beta{+}\beta_0$ are considered and bivariate independent prior density for α and β are used. The problem of solving the Double integral to obtain the closed form for α and β, leads us to use MCMC for calculating the Bayesian Predication Intervals. The use of numerical examples and statistical data has enable to properly present and describe the procedure. We conclude that the Bayesian predication intervals are shorter for y_1 than y_5 when we are increasing the β_0 value.**

Keywords*—Bayesian prediction; Gompertz distribution; predictive distribution; doubly Type-II censored data; Markov Chain Monte Carlo; single outliers

I. Introduction

The adult death patterns can be effectively described through the use of the Gompertz distribution ([17]; [6]). Moreover, the Gompertz mortality force for the decreased infant and young adult levels of mortality extends to the whole life population span without any observed deceleration of mortality ([16]). A continuous probability density function (pdf) and a cumulative distribution function (cdf) are the constituents of the Gompertz distribution.
The pdf as follows:

$$f(x) = \alpha\,\beta\,e^{\alpha\,x - \beta\,(e^{\alpha\,x}-1)}, \quad x > 0, \alpha > 0, \beta > 0, \quad (1)$$

and The cdf as follows:

$$F(x) = 1 - e^{-\beta\,(e^{\alpha\,x}-1)}. \quad (2)$$

This distribution should be denoted with two Gomp α and β parameters. The research conducted by [1] indicated that a simple transformation relates the Gompertz distribution to a certain distribution in the family of distributions. A further research conducted by [7] showed that it is possible to get the maximum likelihood parameter estimates the Gompertz model. The study by [3] suggests the ways to apply it and provides a more recent survey that enables to better understand the model. At the same time, [19] made an attempt to reformulate the Gompertz mortality force and get an insight into the new formation relationship.

The analysis of the research by [18] enabled to trace the connections between the Weibull, the Gompertz, and other Type I extreme value distributions. Later, [9] managed to obtain a Bayesian prediction, mixing two-component lifetime model of Gompertz. In another study [10] derived a Bayesian record statistics analysis from the Gompertz model. A negative Gompertz distribution was presented by later, [11] who focused on the discussion of the negative aging parameter rate. A generalized three-parameter Gompertz distribution was presented by [8]), who provided a deep insight into the topic under investigation. Furthermore, [2]worked on the Gompertz model, and attempted to introduce a more generalized four-parameter version of the model that was referred to as a beta-Gompertz distribution. Also, the paper provides some commonly used distributions, including generalized and beta-exponential Gompertz distributions as sub-models. [15] proposed a distribution of an exponentiated Weibull extension; however, it was modified. It was further generalized and discussed in the study by [8]. Author in [13] focused on the investigation and discussion of the obtained prediction intervals that are based on Gompertz doubly censored data. There are some cases make Progressive Hybrid Censored schemes (PHCS) difficult to apply when the failures may occur before time [21]. Some researchers estimated and predicted the Generalized Progressive Hybrid Censored Data for Gompertz Distribution [20]. Whoever Gompertz distribution was studied by many researchers such as [22].

The main objective of this paper, we assume that $X_I, X_2, \cdots, X_n,$ is an ordered random sample of size n drawn from a population whose pdf, is Gomp(α, β), which is defined by equation 1, and that $Y_1, Y_2, \cdots, Y_m.$ is a second independent random sample (of size m) of future observations from the same distributions. Bayesian prediction bounds for the future observations $Y_t, Y_z, \cdots, Y_m$ in the presence of a single outlier of type $\beta\,\beta_0$ and $\beta + \beta_0$ are obtained.

Observation is an outlier in the data set that is inconsistent with the data set remainder ([5]). Hence, a single $\beta\,\beta_0$, and $\beta + \beta_0$ type outliers are present in the future Gompertz population sample. Gomp$(\alpha,\, \beta\,\beta_0)$ is taken for a single type $\beta + \beta_0$ outlier of the pdf, while in the case of single type $\beta + \beta_0$ outlier the pdf is taken Gomp$(\alpha,\, \beta + \beta_0)$.

In the study, the bounds of the Bayesian prediction are received for the future Gopm $(\alpha,\, \beta)$ distribution observations in the presence of a single outlier of type. It is considered that both parameters α and β are unknown. The true value (β, α) uncertainty is measured through the function of the bivariate prior density that was discussed and applied with the same model in the research conducted by [10].Furthermore, the current research presupposes the construction of the predictive interval that will be used for the future observation with the presence of a single outlier of type with MCMC. The use of statistics will assist in illustrating and presenting the procedure.

In this article, Section II explains the Likelihood Function. After that Section III discuss the Posterior distribution. Moreover, Section IV clarify the Bayesian predication in the presence of outliers for future observations with two schemes $\beta\,\beta_0$ and $\beta + \beta_0$. Section V shows numerical example, which are consider the previous two schemes. In the final Section VI, we give the conclusion and opens future direction.

II. Likelihood Function

In this section, we assume $x_1,\, x_2,\, \cdots,\, x_n$ is an ordered random size n sample from the Gopm$(\alpha,\, \beta)$. The pdf and cdf are given be (1) and (2) , respectively. Also, let $x_1 \leq x_2 \leq \cdots \leq x_k$ be the k smallest ordered observation, while $x_{r+1} \leq x_{r+2} \leq \cdots \leq x_n$, the $n - r$ largest ordered observations in the sample. The statistical analysis contains the application of only the remaining ordered observations, that is, $\underline{x} = (x_{k+1},\, x_{s+2},\, \cdots,\, x_r)$. Moreover, it is evident that when $k = 1$, the sample will be a Type-II right censored sample. A doubly censored sample pulled from population with pdf and cdf as given in (1) and (2) that likelihood function is given as follow:

$$\begin{aligned} L(\alpha, \beta; \underline{x}) \propto\; & [F_X(x_{k+1};\, \alpha,\, \beta)]^k [1 - F_X(x_r;\, \alpha,\, \beta)]^{n-r} \\ & \times \prod_{i=k+1}^{r} [f_X(x_i; \alpha,\, \beta)], x_{s+1} \geq 0 \\ =\; & (\alpha\,\beta)^{r-s} [1 - \exp\{-\beta T_1(\alpha;\, x_{k+1})\}]^k \\ & \times \exp\Big\{\alpha \sum_{i=s+1}^{r} x_i - \beta\, T_2(\underline{x}; \alpha)\Big\}. \end{aligned} \tag{3}$$

where

$$\begin{aligned} T_1(\alpha;\, x_{k+1})) &= e^{\alpha\, x_{k+1}} - 1, \\ T_2(\alpha;\, \underline{x}) &= (n - r) e^{\alpha\, x_r} + \sum_{i=k+1}^{r} e^{\alpha\, x_i} - n + s. \end{aligned} \tag{4}$$

The Bayesian prediction tends to bound the future observations in the presence of a single outlier of type $Gomp(\alpha,\, \beta)$ distribution when two parameters types α and β are both dependent and unknown.

III. The Posterior Distribution

To obtain the joint posterior density of α and β, we use a bivariate prior density of the form:

$$\pi(\alpha,\, \beta) = \pi_1(\alpha)\, \pi_2(\beta), \tag{5}$$

where

$$\pi_1(\alpha) = \frac{\gamma_1^{\eta_1}}{\Gamma(\eta_1)}\, \alpha^{\eta_1 - 1}\, e^{-\alpha \gamma_1},\, (\eta_1,\, \gamma_1 > 0) \tag{6}$$

and

$$\pi_2(\beta) = \frac{\gamma_2^{\eta_2}}{\Gamma(\eta_2)}\, \beta^{\eta_2 - 1}\, e^{-\beta \gamma_2}\, (\eta_2,\, \gamma_2 > 0). \tag{7}$$

The paper assumes that the joint prior density for the parameter α and β is the form (5) and presented by Jaheen [10] for the progressive censored data prediction from the Gompertz model and applied by [13] for the prediction Gompertz doubly censored data intervals.

The likelihood of the function presented by (3) and the function of the joint prior density presented by (5)as well as the function of the joint posterior density of α and β is

$$\pi^*(\alpha,\, \beta|\underline{x}) = \frac{L(\alpha,\, \beta;\, \underline{x}) \pi_1(\alpha)\, \pi_2(\beta)}{\int_0^\infty \int_0^\infty L(\alpha,\, \beta;\, \underline{x}) \pi_1(\alpha)\, \pi_2(\beta) d\alpha d\beta}. \tag{8}$$

The joint posterior density function of α and β given data can be written as

$$\pi^*(\beta,\, \alpha, |\underline{x}) \propto h_1(\beta\,|\,\alpha,\, data) h_2(\alpha|\, data) h_3(\alpha, \beta|\, data) \tag{9}$$

where $h_1(\beta\,|\,\alpha,\, data)$ is a gamma density where the shape parameter $m = r - k + \eta_1$ and the scale parameter is $\gamma_1 + T_2(\alpha;\, \underline{x})$. At the same time, $h_2(\alpha|\, data)$ is a proper density function of the form

$$h_2(\alpha|data) \propto \frac{1}{[\gamma_1 + T_2(\alpha;\, \underline{x})]^m}\, \alpha^{r-k+\eta_2-1}\; e^{-\alpha\,(\frac{1}{\gamma_2} - \sum_{i=k+1}^{r} x_i)} \tag{10}$$

and $h_3(\alpha, \beta|\, data))$ is given by

$$h_3(\alpha,\, \beta|data) = \left[1 - e^{-\beta T_1(\alpha;\, x_{k+1})}\right]^s. \tag{11}$$

From equation (8) and it enables to see that a simple closed form cannot express the equation. Therefore, the Bayes estimators of the parameter α and β cannot be received in simple closed forms. Hence, the paper suggests the approximation (9) by applying the importance sampling technique that is also presented by [14]. The importance sampling details are presented below.

In this paper, we used the importance sampling procedure to calculate the Bayes estimates for α, β as well as any function of the parameters $g(\alpha,\, \beta)$. Moreover, the Algorithm 1 (presented below) is used to generate α and β from the posterior density function (7).

Algorithm 1:

Step 1 : Start with an $(\alpha^0;\, \beta^0)$.

Step 2 : set t = 1.

Step 3 : Generate α^t from $h_2(\alpha|\, data)$ using the method developed by [12] with the $N(\alpha^{t-1},\, \sigma)$ proposal distribution, where σ^2 is the variance of the parameter α.

Step 4 : Generate β^t from gamma distribution with pdf $h_2(\beta\,|\,\alpha,\, data)$.

Step 5 : Put t = t+1.

Step 6 : Repeat steps 3-5 M times to obtain $\{(\alpha^t,\, \beta^t),\, t = 1,\, 2,\, \cdots,\, M\}$.

The approximate Bayes are applied to estimate any function of the parameters say $g(\alpha,\, \beta)$ under the squared functions of error loss using the procedure of importance sampling, as shown below:

$$\hat{g}_{BS}(\alpha,\, \beta) \quad = \frac{\sum_{i=1}^{M} g(\alpha_i,\, \beta_i)\, g_3(\alpha_i,\, \beta_i|data)}{\sum_{i=M_0}^{M} g_3(\alpha_i,\, \beta_i|data)}, \quad (12)$$

IV. Bayesian Prediction in the Presence of a Single Outlier for Future Observations

The section introduces the prediction of the future observations in the presence of a single outlier. Also, it is assumed that $X_1,\, X_2,\, \cdots,\, X_n$ is a random size n sample drawn from the $Gomp(\alpha, \beta)$ population, where the pdf is presented by (1). Let us assume that $Y_1,\, Y_2,\, \cdots,\, Y_m$ is a second, independent, unobserved size m sample received from the same population. This sample is the future sample, and the aim of the study is to get Bayesian prediction bounds for the s^{th} oncoming observation Y_s, $s = 1,\, 2,\, \cdots,\, m$ in the presence of a single outlier.

In the case of the size m sample, let Y_s be the s^{th} ordered lifetime, $1 \leq s \leq m$. Then the Y_s density function for a given θ in the presence of a single outlier is of the form $f = f(y|\,\theta)$ and $F = F(y|\theta)$ are the distribution and density functions of all y_s which are not referred to be outliers as $f^* = f^*(y|\theta)$ and $F^* = F^*(y|\theta)$ are those of an outlier ([4]). The f^* and F^* functions are received for the Gomp$(\alpha,\, \beta)$ model through the replacement of parameter β by $\beta\,\beta_0$ or $\beta + \beta_0$ depending on the outlier type.

$$\begin{aligned} f(y_s|\,\theta) = \;& D(s)\,[(s-1)F^{s-2}(1-F)^{m-s}F^{\star}f \\ & +(m-s)F^{s-1}(1-F)^{m-s-1}(1-F^{\star})f \\ & +F^{s-1}(1-F)^{m-s}f^{\star}], \end{aligned} \quad (13)$$

where

$$D(s) \,=\, \binom{m-1}{s-1} \quad (14)$$

A. Outliers of type $\beta\beta_0$

The Y_s density function, in the presence of a single outlier of type $\beta\beta_0$, in the Gomp$(\alpha,\, \beta)$ case may be received through the substituting of (1) and (2) for f and F in (13). The f^* and F^* values presented by (1) and (2), after the replacement of β by $\beta\beta_0$. It is possible to simplify the density function implementing the pdf $g_1(y_2|\alpha, \beta)$, where the cdf $G_1(y_s|\alpha, \beta)$ is given as follows:

$$\begin{aligned} g_1(y_s|\,\alpha,\, \beta) \,=\, & D(s)\,\alpha\,\beta\, e^{\alpha y_s}\Big[(m+\beta_0-s)\sum_{j=0}^{s-1} A_{1j}(y_s) \\ & + (s-1)\sum_{j=0}^{s-2} A_{2j}(y_s)\Big], \quad y_s \,>\, 0, \end{aligned} \quad (15)$$

where

$$\begin{aligned} A_{1j}(y_s) =\;& a_{1j}(s)\exp\left\{-\beta\omega_j(s)\phi(y_s;\,\alpha)\right\}, \\ A_{2j}(y_s) =\;& a_{2j}(s)\Big[\exp\left\{-\beta\,\omega_{1j}(s)\phi(y_s;\,\alpha)\right\} \\ & -\exp\left\{-\beta\,\omega_{j+1}(s)\phi(y_s;\,\alpha)\right\}\Big], \\ \phi(y_s;\,\alpha) =\;& (e^{\alpha y_s}-1) \\ \omega_j(s) =\;& m\,-\,s\,+\,\beta_0\,+\,j, \\ \omega_{1j}(s) =\;& m\,-\,s\,+\,j\,+\,1 \end{aligned} \quad (16)$$

and for $\ell = 1,\, 2$,

$$a_{\ell\, j}(s) \,=\, (-1)^j\binom{s-\ell}{j}. \quad (17)$$

and the pdf $g_1(y_s|\,\alpha,\, \beta)$ the cdf $G_1(y_s|\,\alpha,\, \beta)$ is given by

$$\begin{aligned} G_1(y_s|\,\alpha,\, \beta) =\, & D(s)\,\Big[(m+\beta_0-s)\sum_{j=0}^{s-1} A^*_{1j}(y_s) \\ & + (s-1)\sum_{j=0}^{s-2} A^*_{2j}(y_s)\Big], \quad y_s \,>\, 0 \end{aligned} \quad (18)$$

where

$$\begin{aligned} A^*_{1j}(y_s) \;=\;& \frac{a_{1j}(s)}{\omega_j(s)}F(y_s;\,\alpha,\,\beta\omega_j(s)), \\ A^*_{2j}(y_s) \;=\;& \frac{a_{2j}(s)}{\omega_{1j}(s)}F(y_s;\,\alpha,\,\beta\omega_{1j}(s)) \\ & -\frac{a_{2j}(s)}{\omega_{j+1}(s)}F(y_s;\,\alpha,\,\beta\omega_{j+1}(s)). \end{aligned} \quad (19)$$

The Bayesian predictive density of y_s, $s = 1,\, 2,\, \cdots,\, m$ given $\underline{x}$ is represented by

$$g_1^*(y_s|\underline{x}) = \int_0^\infty\int_0^\infty g_1(y_s|\,\alpha,\,\beta)\,\pi^*(\alpha,\,\beta|\,\underline{x})\,d\alpha\,d\beta. \quad (20)$$

The Bayesian predictive distribution function of y_s, $s = 1,\, 2,\, \cdots,\, m$ given $\underline{x}$, α and β is given by

$$G_1^*(y_s\,|\underline{x}) = \int_0^\infty\int_0^\infty G_1(y_s|\,\alpha,\,\beta)\,\pi^*(\alpha,\,\beta|\,\underline{x})\,d\alpha\,d\beta. \quad (21)$$

Supposing that $\{(\alpha_i,\, \beta_i);\, i \,=\, 1,\, 2,\, \cdots,M\}$ are MCMC samples received from $\pi^*(\alpha,\, \beta|\,\underline{x})$, it is possible to get the simulation consistent estimators of $g_1^*(y_s|\underline{x})$ and $G^*(y_s|\underline{x})$ can be obtained as

$$\hat{g}_1^*(y_s\,|\underline{x}) \,=\, \sum_{i=1}^{M} g_1(y_s|\,\alpha_i,\,\beta_i)\,h_i \tag{22}$$

and

$$\hat{G}_1^*(y_s\,|\underline{x}) \,=\, \sum_{i=1}^{M} G_1(y_s|\,\alpha_i,\,\beta_i)\,h_i \tag{23}$$

where

$$h_i \,=\, \frac{h_3(\alpha_i,\,\beta_i)}{\sum\limits_{i=1}^{M} h_3(\alpha_i,\,\beta_i)}; \quad i=1,\,2,\,\cdots,\,M. \tag{24}$$

A $(1-\tau)\,100\,\%$ Bayesian prediction interval for Y_s is as follows: $P[L(\underline{x}) \,\le\, Y_s \,\le\, U(\underline{x})] \,=\, 1-\tau$, where $L(\underline{x})$ and $U(\underline{x})$ are the lower and the upper bounds for y_s, $s=1,2,\cdots,m$. Thus, equating of (23) $1-\frac{\tau}{2}$ and $\frac{\tau}{2}$, enables to get the following:

$$P[Y_s \,\ge\, L(\underline{x})|\,\underline{x}] = 1-\frac{\tau}{2} \,\Rightarrow\, \hat{G}_1^*(L(\underline{x})|\,\underline{x}) \,=\, \frac{\tau}{2} \tag{25}$$

and

$$P[Y_s \,\le\, U(\underline{x})|\,\underline{x}] = \frac{\tau}{2} \,\Rightarrow\, \hat{G}_1^*(U(\underline{x})|\,\underline{x}) \,=\, 1-\frac{\tau}{2}. \tag{26}$$

B. Type $\beta+\beta_0$ Outliers

The y_s density function, in the presence of a single outlier of type $\beta+\beta_0$, in the Gomp$(\alpha,\,\beta)$ case, can be received through the substituting of (1) and (2) for F and f in (3). The F^* and f^* are presented by (1) and (2) after the replacement of β by $\beta+\beta_0$. Consequently, the density begins to form:

$$g_2(y_s|\,\alpha,\,\beta) = D(s)\,e^{\alpha\,y_s}\,\Big[(\beta\,(m-s+1)+\beta_0)\sum_{j=0}^{s-1} B_{1j}(y_s) + \beta(s-1)\sum_{j=0}^{s-2} B_{2j}(y_s)\Big],\; y_s>0, \tag{27}$$

where

$$\begin{aligned} B_{1j}(y_s) =&\; a_{1j}(s)\exp\left\{-\left[\beta\,\omega_{1j}(s)+\beta_0\right]\phi(y_s;\,\alpha)\right\} \\ B_{2j}(y_s) =&\; a_{2j}(s)\Big[\exp\left\{-\beta\,\omega_{1j}(s)\phi(y_s;\,\alpha)\right\} \\ &- \exp\left\{-\left[\beta\,\omega_{1(j+1)}(s)+\beta_0\right]\phi(y_s;\,\alpha)\right\}\Big], \end{aligned} \tag{28}$$

$\phi(y_s;\,\alpha)\,\omega_{1j}(s)$ are given in (16) and $a_{\ell j}(s)$, $a_2j(s)$ is given for $\ell=1,\,2$, respectively, by (17).
The cdf corresponding to the pdf $g_2(y_s|\,\alpha,\,\beta)$ is presented by

$$G_2(y_s|\,\alpha,\,\beta) = D(s)\,\Big[(\beta\,(m-s+1)+\beta_0)\sum_{j=0}^{s-1} B_{1j}^*(y_s) + \beta(s-1)\sum_{j=0}^{s-2} B_{2j}^*(y_s)\Big],\; y_s>0, \tag{29}$$

where

$$\begin{aligned} B_{1j}^*(y_s) =&\; \frac{a_{1j}(s)}{\beta\,\omega_{1j}(s)+\beta_0}F(y_s;\,\alpha,\,\beta\,\omega_{1j}(s)+\beta_0), \\ B_{2j}^*(y_s) =&\; \frac{a_{2j}(s)}{\beta\,\omega_{1j}(s)}F(y_s;\,\alpha,\,\beta\,\omega_{1j}(s)) \\ &- \frac{a_{2j}(s)}{\beta\,\omega_{1(j+1)}(s)+\beta_0}F(y_s;\,\alpha,\,\beta\,\omega_{1(j+1)}(s)+\beta_0), \end{aligned} \tag{30}$$

where $F(y_s;\,\alpha,\,\beta\,m+\beta_0)$ is given by(2).

The Bayesian predictive distribution function of y_s, $s=1,\,2,\,\cdots,\,m$ given $\underline{x}$, α and β is given by

$$g_2^*(y_s|\underline{x}) \,=\, \int_0^\infty \int_0^\infty g_2(y_s|\,\alpha,\,\beta)\,\pi^*(\alpha,\,\beta|\,\underline{x})\,d\alpha\,d\beta, \tag{31}$$

and the predictive cdf of y_s, $G_2^*(y_s|\underline{x})$ is given by

$$G_2^*(y_s|\underline{x}) \,=\, \int_0^\infty \int_0^\infty G_2(y_s|\,\alpha,\,\beta)\,\pi^*(\alpha,\,\beta|\,\underline{x})\,d\alpha\,d\beta, \tag{32}$$

where $G_2(y_s|\,\alpha,\,\beta)$ is given by (29) and $\pi^*(\alpha,\,\beta|\,\underline{x})$ is given by (9). It is evident that it is impossible to express (31) and (32) in closed form. Therefore, they cannot be analytically evaluated.
The use of MCMC samples $\{(\alpha_i,\,\beta_i),\,i\,=\,1,\,2,\,\cdots,\,M\}$, enable the obtaining of $g_2^*(y_s|\underline{x})$ and $G_2^*(y_s|\underline{x})$ simulation consistent estimator, as follows:

$$\hat{g}_2^*(y_s|\underline{x}) \,=\, \sum_{i=1}^{M} g_2(y_s|\,\alpha_i,\,\beta_i)\,h_i, \tag{33}$$

and

$$\hat{G}_2^*(y_s|\underline{x}) \,=\, \sum_{i=1}^{M} G_2(y_s|\,\alpha_i,\,\beta_i)\,h_i, \tag{34}$$

Where h_i is given by (24). It is essential to highlight that it is possible to use the same MCMC samples $\{(\alpha_i,\,\beta_i),\,i\,=\,1,\,2,\,\cdots,\,M\}$, to compute $\hat{g}_2^*(y_s|\underline{x})$ and $\hat{G}_2^*(y_s|\underline{x})$ for all y_s. Also, A $(1-\tau)100\%$ Bayesian prediction intervals for is $P[L(\underline{x}) \,\le\, Y_s \,\le\, U(\underline{x})] \,=\, 1-\tau$ where $L(\underline{x})$ and $U(\underline{x})$ are lower and upper y_s Bayesian prediction bounds. Hence, it is possible to get the lower and upper Bayesian prediction bounds, $L(\underline{x})$ and $U(\underline{x})$, for $y_s, s=1,2,\cdot,m$ through solving the following two nonlinear equations.

$$P[Y_s \,\ge\, L(\underline{x})|\,\underline{x}] = 1-\frac{\tau}{2} \,\Rightarrow\, \hat{G}_2^*(L(\underline{x})|\,\underline{x}) \,=\, \frac{\tau}{2} \tag{35}$$

and

$$P[Y_s \,\le\, U(\underline{x})|\,\underline{x}] = \frac{\tau}{2} \,\Rightarrow\, \hat{G}_2^*(U(\underline{x})|\,\underline{x}) \,=\, 1-\frac{\tau}{2}. \tag{36}$$

It is possible to solve the two nonlinear equations (35) and (36) through the use of an iterative method to receive the lower and upper Bayesian prediction bounds for $y_s; s=1,2,\cdots,m$.

V. Numerical Example

Example 1. This example shows a doubly Type-II censored sample, $x_{(s+1)}, x_{(s+2)}, \cdots, x_{(r)}$, that is received through the application of the following steps:

1 – For the hyperparameters given values $\eta_1 = 1.2$ and $\gamma_1 = 1.8$ a generated value of $\alpha = 0.860986$ is received from the prior distribution with pdf (6).

2 – For the hyperparameters given values $\eta_2 = 1.4$ and $\gamma_2 = 1.7$ a generated value of $\beta = 0.409442$ is received from the prior distribution with pdf (7).

3 – The use of the generated values of α and β from two prior steps, enables to generate a sample of size $n = 30$ from the Gomp(α, β) distribution with pdf, that is represented by (2).

4 – The application of some sorting routine, assists in obtaining a doubly Type-II censored different value sample of size $r = 20, 25, 30$ and $k = 0, 5, 10$ from the Gomp(α, β) distribution, where the deferment value of r and k is presented in Tables I, II and III.

5 – Generate (α_i, β_i), $i = 1, 2, \cdots, M$, through the use of MCMC shown in Algorithm 1.

6 – The above generated doubly Type-II censored size $(r-s)$ sample, the 95 % Bayesian prediction links to the future ordered values, $y_{(1)}, y_{(2)}, \cdots, y_{(m)}$, $m = 5$ in the single types $\beta\,\beta0$ outliers, enable a numerical calculation through solving the equations (25) and (26).

Let us assume that we have one more size $m = 5$ sample in the presence of a single outlier of type $\beta\,\beta_0$. Hence, for the given β_0 values we seek to receive 95% Bayesian prediction bounds for y_1 to y_5 of the failure future sample times. Tables I, II and III represents these bounds with the corresponding β_0 values.

Example 2. The 95% Bayesian prediction interval for a future unobserved y_1 to y_5, which are the failure times in the future size 5 sample in the presence of a single outlier of type $\beta + \beta_0$ can be obtained on the basis of a generated doubly Type-II censored sample of size$n$ from the Gomp (α, β) distribution. Same different $\eta_1, \gamma_1, \eta_2, \gamma_1$ hyper-parameter values and the same data set is presented in Example 1. Hence, these bounds with the corresponding $n = 30$, $r = 20, 25, 20$ and $k = n-r$ and β_0 values are shown in Tables IV, V and VI.

VI. Conclusion

The study investigated and discussed the single $\beta\,\beta_0$ and $\beta + \beta_0$ type outliers through the application of the predictive distribution function. Hence, the Bayesian prediction intervals in the case of future homogeneous case observations can be received by $\beta_0 = 1$ in (18) or $\beta_0 = 0$ in (29).
However, it is impossible in the no outlier case. The Gibbs sampling technique was applied to generate MCMC samples. Afterwards, the importance sampling methodology was used to compute the Bayesian prediction problems in the presence of a single outlier of both type. It is essential to highlight that the Bayesian prediction intervals are shorter for y_1 and larger for the Bayesian prediction intervals for y_5 due to the increase of β_0 value.

TABLE I. 95 % Bayesian prediction intervals for $y_1, \cdots, y_5$ in the presence of a single outlier of type $\beta\,\beta_0$, where $n = 30$, $r = 20$, $k = 10$. Note: Obs. is observations PP is point predictors, LB is Lower Bound, UB is Upper Bound, CP is Coverage Percentages.

β_0	Obs	y_1	y_2	y_3	y_4	y_5
1	PP	0.492721	0.933387	1.35569	1.80134	2.37285
	LB	0.017663	0.177097	0.460092	0.824819	1.29551
	UB	1.39901	1.88431	2.30933	2.76994	3.43655
	Length	1.38135	1.70721	1.84924	1.94513	2.14104
	CP	95.77 %	95.62 %	95.03 %	94.79 %	93.86 %
2	PP	0.428709	0.836901	1.24704	1.69552	2.285
	LB	0.014736	0.151347	0.403924	0.743443	1.19871
	UB	1.2507	1.73927	2.1873	2.68116	3.38701
	Length	1.23597	1.58792	1.78338	1.93772	2.1883
	CP	95.17 %	95.84 %	95.91 %	96.12 %	95.65 %
3	PP	0.379907	0.77606	1.19357	1.65873	2.26774
	LB	0.012642	0.13457	0.370458	0.700365	1.15708
	UB	1.13273	1.66262	2.15497	2.67353	3.38666
	Length	1.12009	1.52805	1.78451	1.97316	2.22958
	CP	94.19 %	95.73 %	96.4%	96.77 %	96.31 %
4	PP	0.34137	0.735004	1.16389	1.64286	2.26251
	LB	0.011069	0.122535	0.34753	0.6731	1.13514
	UB	1.03625	1.62423	2.14778	2.67299	3.38665
	Length	1.02518	1.5017	1.80025	1.99989	2.25152
	CP	92.88 %	95.61 %	96.64 %	97.25 %	96.6 %
5	PP	0.310111	0.705889	1.14598	1.63502	2.2605
	LB	0.009844	0.113362	0.33055	0.654265	1.12263
	UB	0.955644	1.60625	2.14633	2.67296	3.38665
	Length	0.945801	1.49289	1.81578	2.01869	2.26402
	CP	91.31 %	95.49 %	96.86 %	97.55 %	96.77 %

TABLE II. 95 % Bayesian prediction intervals for $y_1, \cdots, y_5$ in the presence of a single outlier of type $\beta\,\beta_0$, where $n = 30$, $r = 25$, $k = 5$. Note:Obs. is observations PP is point predictors, LB is Lower Bound, UB is Upper Bound, CP is Coverage Percentages.

β_0	Obs	y_1	y_2	y_3	y_4	y_5
1	PP	0.480366	0.912016	1.3272	1.76659	2.33151
	LB	0.017118	0.171888	0.447624	0.804559	1.26715
	UB	1.3691	1.84794	2.26819	2.72426	3.38507
	Length	1.35198	1.67605	1.82057	1.9197	2.11793
	CP	95.69 %	95.69 %	95.12 %	95.04 %	94.49 %
2	PP	0.417723	0.817269	1.22021	1.66216	2.24463
	LB	0.014281	0.146862	0.392804	0.724791	1.17186
	UB	1.22303	1.7047	2.14747	2.63631	3.33594
	Length	1.20875	1.55784	1.75466	1.91152	2.16407
	CP	94.88 %	95.76 %	95.86 %	96.25 %	96.03 %
3	PP	0.370002	0.757583	1.16762	1.6259	2.22759
	LB	0.012251	0.130561	0.360161	0.682593	1.13091
	UB	1.10696	1.62904	2.11549	2.62875	3.33559
	Length	1.09471	1.49848	1.75533	1.94616	2.20467
	CP	94. %	95.41 %	96.21 %	96.84 %	96.5 %
4	PP	0.332341	0.717339	1.13846	1.61028	2.22243
	LB	0.010726	0.118871	0.337806	0.655895	1.10933
	UB	1.01211	1.59117	2.10838	2.62822	3.33558
	Length	1.00139	1.4723	1.77057	1.97233	2.22625
	CP	92.5 %	95.15 %	96.33 %	97.33 %	96.79 %
5	PP	0.30181	0.688818	1.12087	1.60256	2.22046
	LB	0.009539	0.109964	0.321256	0.637456	1.09703
	UB	0.93294	1.57343	2.10694	2.62818	3.33558
	Length	0.923401	1.46347	1.78568	1.99073	2.23855
	CP	90.53 %	95.11 %	96.51 %	97.58 %	96.94 %

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TABLE III. 95 % BAYESIAN PREDICTION INTERVALS FOR $y_1, \cdots, y_5$ IN THE PRESENCE OF A SINGLE OUTLIER OF TYPE $\beta\,\beta_0$, WHERE $n = 30$, $r = 30$, $k = 0$. NOTE:OBS. IS OBSERVATIONS PP IS POINT PREDICTORS, LB IS LOWER BOUND, UB IS UPPER BOUND, CP IS COVERAGE PERCENTAGES.

β_0	Obs	y_1	y_2	y_3	y_4	y_5
1	PP	0.475422	0.902867	1.31418	1.74964	2.30966
	LB	0.016929	0.170029	0.442908	0.796326	1.25459
	UB	1.35561	1.8302	2.24682	2.69903	3.35434
	Length	1.33868	1.66017	1.80392	1.9027	2.09975
	CP	95.56 %	95.69 %	95.21 %	95.11 %	94.63 %
2	PP	0.413396	0.809015	1.20817	1.64613	2.22353
	LB	0.014124	0.145269	0.388644	0.717329	1.16018
	UB	1.21088	1.68822	2.12713	2.61182	3.30561
	Length	1.19675	1.54295	1.73849	1.89449	2.14543
	CP	94.81 %	95.63 %	95.72 %	96.19 %	95.98 %
3	PP	0.366148	0.7499	1.15607	1.61019	2.20664
	LB	0.012116	0.129143	0.356335	0.675541	1.11961
	UB	1.09588	1.61323	2.09543	2.60433	3.30526
	Length	1.08376	1.48409	1.73909	1.92878	2.18565
	CP	93.81 %	95.25 %	95.95 %	96.84 %	96.49 %
4	PP	0.328865	0.710044	1.12718	1.59472	2.20152
	LB	0.010608	0.117579	0.334211	0.649105	1.09822
	UB	1.00191	1.57569	2.08838	2.6038	3.30526
	Length	0.991305	1.45811	1.75417	1.95469	2.20703
	CP	92.35 %	94.94 %	96.15 %	97.25 %	96.81 %
54	PP	0.298642	0.681802	1.10976	1.58707	2.19956
	LB	0.009434	0.108767	0.317832	0.630847	1.08604
	UB	0.923483	1.55811	2.08696	2.60376	3.30526
	Length	0.914049	1.44934	1.76912	1.97292	2.21922
	CP	90.22 %	94.88 %	96.31 %	97.43 %	96.93 %

TABLE IV. 95 % BAYESIAN PREDICTION INTERVALS FOR $y_1, \cdots, y_5$ IN THE PRESENCE OF A SINGLE OUTLIER OF TYPE $\beta + \beta_0$, WHERE $n = 30$, $r = 20$, $k = 10$. NOTE: OBS. IS OBSERVATIONS PP IS POINT PREDICTORS, LB IS LOWER BOUND, UB IS UPPER BOUND, CP IS COVERAGE PERCENTAGES.

β_0	Obs	y_1	y_2	y_3	y_4	y_5
0	PP	0.314436	0.624792	0.947281	1.3117	1.80927
	LB	0.010012	0.103158	0.280504	0.530439	0.8847
	UB	0.966839	1.36921	1.7411	2.15974	2.78612
	Length	0.956827	1.26605	1.46059	1.6293	1.90142
	CP	95.48 %	95.68 %	95.32 %	95.39 %	95.04 %
1	PP	0.249322	0.524206	0.833376	1.20209	1.72088
	LB	0.00759	0.081351	0.230135	0.452727	0.787286
	UB	0.792205	1.20258	1.61103	2.07199	2.7386
	Length	0.784615	1.12123	1.3809	1.61926	1.95131
	CP	94.01 %	94.96 %	95.6 %	96.61 %	96.6 %
2	PP	0.206843	0.474838	0.794469	1.17926	1.71243
	LB	0.006111	0.069507	0.205725	0.420648	0.757922
	UB	0.67229	1.14618	1.59745	2.07048	2.73858
	Length	0.666179	1.07668	1.39172	1.64983	1.98066
	CP	91.36%	94.35%	95.9 %	97.18 %	97.02 %
3	PP	0.176862	0.446821	0.777449	1.17201	1.71067
	LB	0.005115	0.06181	0.190618	0.402904	0.746022
	UB	0.584482	1.12971	1.59639	2.07045	2.73858
	Length	0.579367	1.0679	1.40577	1.66755	1.99256
	CP	87.7%	94.34%	96.26%	97.31%	97.2%
4	PP	0.154538	0.429346	0.768791	1.16911	1.71014
	LB	0.004398	0.056299	0.180133	0.391883	0.740829
	UB	0.517254	1.12552	1.59631	2.07045	2.73858
	Length	0.512856	1.06922	1.41618	1.67857	1.99775
	CP	84.05%	94.45%	96.45%	97.45%	97.28 %

TABLE V. 95 % BAYESIAN PREDICTION INTERVALS FOR $y_1, \cdots, y_5$ IN THE PRESENCE OF A SINGLE OUTLIER OF TYPE $\beta + \beta_0$, WHERE $n = 30$, $r = 25$, $k = 5$. NOTE: OBS. IS OBSERVATIONS PP IS POINT PREDICTORS, LB IS LOWER BOUND, UB IS UPPER BOUND, CP IS COVERAGE PERCENTAGES.

β_0	Obs	y_1	y_2	y_3	y_4	y_5
0	PP	0.310392	0.617041	0.935936	1.29655	1.78929
	LB	0.009873	0.101746	0.276772	0.523637	0.873873
	UB	0.955124	1.35334	1.72163	2.1364	2.75727
	Length	0.945251	1.25159	1.44485	1.61277	1.88339
	CP	95.48%	95.69%	95.34%	95.48%	95.16%
1	PP	0.246376	0.518007	0.823625	1.18832	1.70186
	LB	0.007495	0.080324	0.227241	0.447124	0.777809
	UB	0.783238	1.18895	1.59301	2.04949	2.71016
	Length	0.775743	1.10863	1.36578	1.60236	1.93235
	CP	93.86%	94.87%	95.47%	96.52%	96.53%
2	PP	0.204538	0.469253	0.785086	1.16562	1.69343
	LB	0.00604	0.06866	0.203175	0.41544	0.748708
	UB	0.665036	1.13293	1.57938	2.04795	2.71014
	Length	0.658996	1.06427	1.37621	1.63251	1.96143
	CP	91.04 %	94.21%	95.75%	97.01%	96.97 %
3	PP	0.174974	0.441532	0.768182	1.15839	1.69166
	LB	0.005059	0.061072	0.188266	0.397885	0.736867
	UB	0.578397	1.11642	1.5783	2.04793	2.71014
	Length	0.573339	1.05535	1.39004	1.65004	1.97327
	CP	87.33%	94.02%	96.06%	97.16%	97.14%
4	PP	0.152942	0.424219	0.759568	1.15549	1.69113
	LB	0.004351	0.055635	0.17791	0.386966	0.731677
	UB	0.512017	1.11217	1.57822	2.04793	2.71014
	Length	0.507666	1.05654	1.40031	1.66096	1.97846
	CP	3.69%	94.13%	96.27%	97.27%	97.24 %

TABLE VI. 95 % BAYESIAN PREDICTION INTERVALS FOR $y_1, \cdots, y_5$ IN THE PRESENCE OF A SINGLE OUTLIER OF TYPE $\beta + \beta_0$, WHERE $n = 30$, $r = 30$, $k = 0$. NOTE: OBS. IS OBSERVATIONS PP IS POINT PREDICTORS, LB IS LOWER BOUND, UB IS UPPER BOUND, CP IS COVERAGE PERCENTAGES.

β_0	Obs	y_1	y_2	y_3	y_4	y_5
0	PP	0.309735	0.615638	0.933667	1.29322	1.78437
	LB	0.009855	0.10156	0.276229	0.522522	0.871834
	UB	0.952855	1.34988	1.71698	2.13035	2.74903
	Length	0.943	1.24831	1.44075	1.60783	1.8772
	CP	95.48 %	95.67 %	95.32 %	95.46 %	95.14 %
1	PP	0.245765	0.516724	0.821549	1.18522	1.69719
	LB	0.007478	0.080147	0.226737	0.446102	0.77594
	UB	0.781161	1.18581	1.58871	2.04372	2.70209
	Length	0.773683	1.10566	1.36198	1.59762	1.92615
	CP	93.81%	94.77%	95.43%	96.46%	96.53%
2	PP	0.203983	0.46808	0.783138	1.16263	1.68881
	LB	0.006025	0.068498	0.202712	0.414491	0.74694
	UB	0.66315	1.13002	1.57519	2.0422	2.70207
	Length	0.657125	1.06152	1.37248	1.62771	1.95513
	CP	90.95%	94.14%	95.69%	96.96%	96.95%
3	PP	0.174471	0.44044	0.766305	1.15544	1.68705
	LB	0.005045	0.060923	0.187833	0.396987	0.735156
	UB	0.576681	1.11363	1.57412	2.04218	2.70207
	Length	0.571636	1.05271	1.38629	1.6452	1.96692
	CP	87.27%	93.94%	96.01%	97.09%	97.13%
4	PP	0.152484	0.423186	0.757732	1.15256	1.68652
	LB	0.004339	0.055497	0.177501	0.386105	0.729999
	UB	0.510446	1.10943	1.57404	2.04218	2.70207
	Length	0.506108	1.05393	1.39654	1.65608	1.97207
	CP	83.63%	94.05%	96.22%	97.2%	97.23%

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