Super-Twisting Sliding Mode Distributed Consensus for Nonlinear Multi-Agent Systems with Unknown Bounded External Disturbances

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***Abstract*—This paper addresses the distributed consensus tracking problem for nonlinear multi-agent systems subject to unknown but bounded external disturbances by leveraging a super-twisting sliding mode (STSM) control framework. Two STSM-based consensus algorithms are proposed—one for first-order and another for second-order multi-agent systems—to achieve finite-time convergence despite disturbances. A disturbance observer is integrated into the consensus control protocols to estimate and compensate for these disturbances, ensuring robust tracking without requiring time-derivative sliding variables or smoothing algorithms. The proposed consensus protocols build upon the concepts of finite-time stability, Lipschitz-bounded functions, relative degree analysis of input-output dynamics, and positive-definite matrix properties. Stability and finite-time convergence are rigorously established using Lyapunov-based proofs, Rayleigh's inequality, and finite-time settling results. Unstructured disturbances are modelled as zero-mean Gaussian noise and structured disturbances are expressed via a regressor formulation. Numerical simulations confirm that the integrated STSM-based consensus approach and disturbance observer ensure high tracking accuracy, robustness, and smooth control performance under diverse disturbance conditions.**

Keywords*—Distributed consensus; cooperative control; nonlinear multiagent systems; robustness; super-twisting sliding mode

I. Introduction

Distributed consensus control has emerged as a fundamental approach for coordinating multi-agent systems (MAS), enabling agents to achieve a common goal through local interactions [1]. This decentralized control paradigm has been widely applied in robotics, unmanned aerial vehicles (UAVs), distributed sensor networks, and intelligent transportation systems due to its scalability and robustness against single-point failures [2]. Traditional consensus algorithms rely on linear or adaptive control techniques to ensure convergence; however, external disturbances, model uncertainties, and time-varying perturbations significantly complicate the consensus process [3]. To address these challenges, sliding mode control (SMC) has been extensively adopted for MAS coordination due to its inherent robustness against disturbances and uncertainties [4]. First-order sliding-mode (FOSM) control has been widely implemented to counteract local interaction uncertainties and external perturbations [5]. However, a well-known drawback of FOSM is the chattering phenomenon, which can lead to excessive energy consumption, actuator degradation, and performance deterioration in practical applications [6]. Various mitigation strategies, such as boundary layers [7], saturation control [8], and adaptive filtering techniques [9], have been proposed to alleviate chattering, but these methods often introduce a tradeoff between robustness and precision.

Recent advancements in high-order sliding-mode control (HOSM) have significantly improved the performance of SMC-based consensus algorithms. Among these, super-twisting sliding-mode control (STSMC) has gained substantial attention due to its ability to suppress chattering while preserving finite-time convergence and disturbance rejection capabilities [10]. STSMC introduces a continuous control law that effectively reduces oscillations near the sliding manifold while maintaining the robustness of conventional sliding-mode strategies. Numerous studies have explored the application of STSMC in MAS, demonstrating its effectiveness in various scenarios. For instance, Song, Yu, and Zheng [11] developed an STSMC-based consensus tracking algorithm that guarantees finite-time convergence under bounded disturbances. Similarly, Li, Wang, and Zhang [12] extended STSMC to distributed control frameworks, explicitly addressing time-varying uncertainties and ensuring robust coordination in uncertain environments. Additionally, Wang, Chou, and Liu [13] proposed adaptive STSMC strategies to handle leader-follower MAS with parametric uncertainties. Zhang, Liu, and Song [14] implemented STSMC-based formation control techniques for UAVs subjected to aerodynamic disturbances and dynamic payload variations.

Beyond traditional consensus tracking, researchers have proposed observer-based STSMC approaches to accommodate cases where state measurements are unavailable or incomplete. Authors in [15] introduced an observer-based STSMC method to estimate unmeasured states in uncertain MAS, enhancing the robustness of the control strategy. In [16] authors developed output-feedback STSMC techniques to handle stochastic disturbances and measurement noise, further improving the resilience of distributed consensus protocols. In addition, event-triggered STSMC methodologies have been introduced to reduce communication overhead in resource-constrained MAS networks by ensuring that control updates are executed only when necessary [17]. Despite these advancements, the most existing STSMC-based consensus control strategies assume that disturbances are either fully known or follow a predefined model, which is rarely the case in real-world applications [18].

In practical settings, disturbances often arise from unpredictable environmental changes, sensor noise, actuation delays, and communication constraints, making it imperative to develop control strategies capable of real-time disturbance estimation and rejection.

The primary challenge addressed in this study is developing a robust STSMC-based consensus control framework that actively estimates and rejects unknown bounded external disturbances in MAS. Conventional STSMC techniques, while effective in suppressing chattering and enhancing robustness, do not inherently incorporate mechanisms for real-time disturbance adaptation [19]. This limitation necessitates conservative gain tuning, which can lead to sluggish transient responses and reduced disturbance rejection efficiency. By integrating structured disturbance observers into the STSMC framework, this work aims to achieve real-time estimation of unknown disturbances, thereby improving the controller's adaptability and overall performance [20]. The proposed approach ensures that agents within the MAS can maintain finite-time consensus tracking despite external uncertainties while mitigating excessive control effort and minimizing chattering effects.

This research addresses key questions regarding the design and implementation of distributed STSMC for nonlinear MAS under unknown bound disturbances. Specifically, it investigates how distributed STSMC can be structured to achieve robust finite-time consensus tracking under uncertain disturbances. Additionally, it explores which disturbance estimation techniques can be effectively integrated into the STSMC framework to enhance disturbance rejection without compromising chattering suppression. Furthermore, this study evaluates the proposed method's performance relative to conventional FOSM, STSMC, and adaptive control strategies, considering convergence speed, robustness, and control effort metrics.

To address these research challenges, this work presents two main contributions. First, it develops a novel STSMC-based distributed consensus-tracking algorithm tailored for first-order and second-order nonlinear MAS. This algorithm ensures that consensus is reached in finite time while actively rejecting external disturbances through an embedded disturbance observer. Second, it establishes rigorous theoretical guarantees for stability and robustness, proving that the proposed approach maintains finite-time convergence under a general class of bounded disturbances. These advancements aim to bridge the gap in STSMC-based consensus control by enabling real-time disturbance adaptation without sacrificing robustness or performance.

The effectiveness of the proposed method is validated through extensive numerical simulations, where its performance is compared against existing sliding-mode and adaptive consensus control techniques. The simulations analyze key performance indicators such as tracking error convergence, disturbance rejection efficiency, and chattering suppression. The results demonstrate that the proposed STSMC approach significantly improves disturbance handling and consensus tracking precision while reducing unnecessary control effort. These findings indicate that integrating structured disturbance observers into STSMC provides a practical and scalable solution for MAS applications operating in uncertain and dynamically evolving environments.

In the context of MAS, recent studies have explored various control strategies to enhance coordination and performance. For instance, authors in [21] proposed a distributed cooperative control framework for multi-UAV flying formations, addressing challenges such as chattering effects and formation tracking in three-dimensional space. Their approach integrates smooth control protocols within a leader-following framework, ensuring robust formation maintenance despite external disturbances and communication constraints. Similarly, in the realm of multi-robot systems, authors in [22] developed a distributed cooperative control strategy for nonholonomic wheeled mobile robots, focusing on smooth consensus protocols to improve coordination and reduce chattering phenomena. In satellite formation flying, a distributed attitude synchronization control method for switched networked satellite formations was introduced in [23] ensuring finite-time convergence and robustness against switching topologies and external disturbances. These contributions collectively advance the field of distributed control in MAS, offering practical solutions for complex aerospace and robotic applications.

The remainder of this paper is structured as follows: Section 2 presents preliminaries of distributed consensus and coordinated control. The consensus tracking problem for first-order and second-order dynamic MAS including disturbance observer is formulated and solved in section 3 and section 4, respectively. Section 5 validates the effectiveness of the proposed approach through numerical simulations and comparative studies. Finally, Section 6 concludes the paper with key findings, potential limitations, and future research directions.

II. PRELIMINARIES

A. Graph Theory and Preliminaries

Consider the case of MAS composed of n agents connected under a communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ of order n, where $\mathcal{V} = (v_1, v_2, \ldots, v_n)$, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and $\mathcal{A} = (a_{ij}) \in \mathbb{R}^{n \times n}$ are the node set, edge set, and weighted adjacency matrix, respectively.

Assumption 1. A Laplacian matrix $\boldsymbol{\mathcal{L}}$ is associated with the graph $\mathcal{G}$ such that $\boldsymbol{\mathcal{L}} = [l_{ij}]) \in \mathbb{R}^{n \times n}$ where $l_{ij} = -a_{ij}$ when $i \neq j$ and $l_{ii} = \sum_{j=1, j \neq i}^{n} a_{ij}$.

Assumption 2. The graph $\mathcal{G}$ is connected and the eigenvalues $\lambda_i(\boldsymbol{\mathcal{L}})$ of the Laplacian matrix $\boldsymbol{\mathcal{L}}$ are defined such that $\lambda_1(\boldsymbol{\mathcal{L}}) = 0 < \lambda_2(\boldsymbol{\mathcal{L}}) <\ldots< \lambda_n(\boldsymbol{\mathcal{L}})$. $\lambda_1(\boldsymbol{\mathcal{L}}) = 0$ has an associated eigenvector **1**.

Assumption 3. There exists a symmetric positive definite matrix $\boldsymbol{M}$ such that $\boldsymbol{M} = \boldsymbol{\mathcal{L}} + \boldsymbol{diag}(a_{10}, a_{20}, \ldots, a_{n0})$.

Lemma 1 (Rayleigh's inequality, Horn and Johnson, 1986)**.** If a matrix $\boldsymbol{Q}$ is symmetric $\boldsymbol{Q} = \boldsymbol{Q}^T$, then for a given bounded vector $\boldsymbol{v}$

$$\lambda_{min}(\boldsymbol{Q}) \|\boldsymbol{v}\|^2 \leq \boldsymbol{v}^T \boldsymbol{Q} \boldsymbol{v} \leq \lambda_{max}(\boldsymbol{Q}) \|\boldsymbol{v}\|^2 \tag{1}$$

where λ_{min} and λ_{max} are the minimum and maximum eigenvalues of $\boldsymbol{Q}$, respectively.

B. Second-Order Super-Twisting Sliding Mode

Consider a *m*-order SISO nonlinear dynamic system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})u \\ \sigma = \sigma(\boldsymbol{x}) \tag{2}$$

where $\boldsymbol{x} \in \mathbb{R}^m$ is the system state and $u \in \mathbb{R}$ is the control input; $\boldsymbol{f} \in \mathbb{R}^m$ and $\boldsymbol{g} \in \mathbb{R}^m$ are uncertain smooth functions; σ is the tracking error (sliding variable).

The control objective of the second-order STSM control is to exactly stabilize $\sigma(\boldsymbol{x})$ and its first time derivative $\dot{\sigma}(\boldsymbol{x})$ in finite time without the use of $\dot{\sigma}(\boldsymbol{x})$ and without affecting the tracking performance. The control task is to drive the system trajectories to reach $\sigma(\boldsymbol{x}) = \dot{\sigma}(\boldsymbol{x}) = 0$ in finite time. The STSM control law is designed under the following assumptions.

Assumption 4. The relative degree of the input-output dynamics $u \to \sigma$ is one and the internal dynamics are stable

$$\dot{\sigma}(\boldsymbol{x}) = \eta(\boldsymbol{x}) + \zeta(\boldsymbol{x})u \tag{3}$$

with $\eta(x) = \dot{\sigma}(x)|_{u=0}$ and $\zeta(\boldsymbol{x}) = \partial\sigma(\boldsymbol{x})/\partial u \neq 0$

Definition 1. The system (2) is said to be a finite-time stable system in a compact $\boldsymbol{X} \subset \mathbb{R}^m$ if, $\forall \boldsymbol{x}_0 \in \boldsymbol{X}$, the system is asymptotically stable with a finite time settling for any solution $\boldsymbol{x}$ (see Bhatt & Bernstein, 2000; Bacciotti & Rosier, 2005).

Lemma 2 [17]: for any Lipschitz bounded function $\boldsymbol{f}$, there exists a constant $p \geq 2$ and positive gains K_1 and K_2 for which a finite-time convergence $\sigma(\boldsymbol{x}), \dot{\sigma}(\boldsymbol{x}) \to 0$ can be provided by the following STSM control law without the usage of $\dot{\sigma}(\boldsymbol{x})$

$$u(\boldsymbol{x}) = -K_1|\sigma(\boldsymbol{x})|^{\frac{p-1}{p}} sign\big(\sigma(\boldsymbol{x})\big) + v(\boldsymbol{x}) \\ \dot{v}(\boldsymbol{x}) = -K_2|\sigma(\boldsymbol{x})|^{\frac{p-2}{p}} sign\big(\sigma(\boldsymbol{x})\big) \tag{4}$$

where $v(\boldsymbol{x})$ is the controller state.

III. Consensus-Tracking for First-Order Dynamics

A. Problem Statement

Consider a class of first-order MAS composed of one virtual leader (labelled as *0*) and '*n'* identical physical followers (labelled agent *i* with *i = 1, n*) described by the following first-order nonlinear uncertain dynamics subject to unknown bounded disturbances. The leader's dynamics are*:*

$$\dot{\boldsymbol{x}}_0 = \boldsymbol{f}_0(\boldsymbol{x}_0),\ \boldsymbol{y}_0 = \boldsymbol{h}_0(\boldsymbol{x}_0) \tag{5}$$

where $\boldsymbol{x}_0 \in \mathbb{R}^m$ and $\boldsymbol{y}_0 \in \mathbb{R}^q$ are the leader's state and output, respectively. The vector-valued functions $\boldsymbol{f}_0 \in \mathbb{R}^m$ and $\boldsymbol{h}_0 \in \mathbb{R}^q$ are continuous functions that describe a leader's dynamics and response, respectively. The followers' dynamics are

$$\dot{\boldsymbol{x}}_i = \boldsymbol{f}_i(\boldsymbol{x}_i) + \boldsymbol{G}_i(\boldsymbol{x}_i, \boldsymbol{u}_i)[\boldsymbol{u}_i(\boldsymbol{x}_i) + \boldsymbol{d}_i(\boldsymbol{x}_i, t)]\boldsymbol{y}_i = \boldsymbol{h}_i(\boldsymbol{x}_i) \tag{6}$$

where $\boldsymbol{x}_i \in \mathbb{R}^m$, $\boldsymbol{u}_i \in \mathbb{R}^m$, $\boldsymbol{y}_i \in \mathbb{R}^q$, and $\boldsymbol{d}_i \in \mathbb{R}^m$ are the i^{th} follower's state, control input, output, and disturbance vectors, respectively. The vector-valued functions $\boldsymbol{f}_i \in \mathbb{R}^m$ and $\boldsymbol{h}_i \in \mathbb{R}^q$ are uncertain continuous functions that describe the follower's dynamics and responses, respectively. In this study, we consider only the case of affine control inputs with $\boldsymbol{G}_i \equiv \boldsymbol{I}_m$, $\boldsymbol{y}_i = \boldsymbol{x}_i$, and $\boldsymbol{y}_0 = \boldsymbol{x}_0$.

Assumption 5. For each agent '*i*', the uncertainties/disturbances $\boldsymbol{d}_i(\boldsymbol{x}_i, t)$ are Lipschitz-continuous functions growing in time and/or with state variables and are bounded such that

$$\lim_{t\to\infty} |\boldsymbol{d}_i(\boldsymbol{x}_i, t)| = \boldsymbol{\zeta}_i \tag{7}$$

where $\zeta_i \in \mathbb{R}^+$.

The problem addressed in this section consists of finding smooth control inputs $\boldsymbol{u}_i(\boldsymbol{x}_i)$ to enforce the followers' kinematics (6) reaching the following consensus condition robustly

$$\lim_{t\to T} \|\boldsymbol{x}_i(t) - \boldsymbol{x}_0(t)\|_\infty = 0 \qquad \forall i = 1,2,\ldots,n \tag{8}$$

To achieve the main results of robust distributed consensus protocols, we define a tracking variable σ, for each follower '*i=1,n*' and along each motion direction '*k=1,m*', as follows

$$\sigma_{i,k}(\boldsymbol{x}_i) = \textstyle\sum_{j=0}^{n} a_{ij}\big(x_{i,k} - x_{j,k}\big) \tag{9}$$

Assumption 6: The relative degree of the sliding variables $\sigma_{i,k}$ concerning the control inputs $u_{i,k}$ is one, for which the desired consensus (8) is achieved when $\sigma_{i,k} \equiv 0$ and the associated internal dynamics are stable.

The distributed consensus-tracking algorithm is designed such that the protocols $u_{i,k}$ ensure that the kinematics of the follower '*i*' robustly track the ones of the virtual leader with local interaction in the presence of matched disturbances. We propose a new variant of the Lyapunov-based STSM control law (4)

$$u_{i,k} = -K_1\|\boldsymbol{\sigma}_k(\boldsymbol{x}_i)\|_\infty^{\frac{p-1}{p}} sign\left(\sigma_{i,k}(\boldsymbol{x}_i)\right) + v_{i,k} + \hat{d}_{i,k}(\boldsymbol{x}_i) \\ \dot{v}_{i,k} = -K_2\|\boldsymbol{\sigma}_k(\boldsymbol{x}_i)\|_\infty^{\frac{p-2}{p}} sign\left(\sigma_{i,k}(\boldsymbol{x}_i)\right) \tag{10}$$

where $\|\boldsymbol{\sigma}_k(\boldsymbol{x}_i)\|_\infty$ defines the infinity norm of the sliding vector $\boldsymbol{\sigma}_k(\boldsymbol{x}) = \left[\sigma_{1,k}(\boldsymbol{x}_i), \ldots, \sigma_{m,k}(\boldsymbol{x}_i)\right]^T$ along the motion direction '*k*' and $\hat{d}_{i,k}$ are the estimated values of the disturbances $d_{i,k}$, to be estimated through special observers to be developed further.

B. Unperturbed Dynamics

Consider the MAS (5)-(6) in its nominal form (i.e. without uncertainties and/or disturbances). Let $\tilde{\boldsymbol{x}}_i = \boldsymbol{x}_i - \boldsymbol{x}_0 \in \mathbb{R}^m$ being the consensus state error vector, we rewrite the dynamics (6), for the unperturbed case, as

$$\dot{\tilde{\boldsymbol{x}}}_i = \boldsymbol{f}_i(\boldsymbol{x}_i) - \boldsymbol{f}_0(\boldsymbol{x}_0) + \boldsymbol{u}_i(\tilde{\boldsymbol{x}}_i) \tag{11}$$

Using the STSMC law (10) with $\boldsymbol{d}_i = 0$, the consensus dynamics (11) can be written in matrix form as

$$\dot{\boldsymbol{e}} = \boldsymbol{F}(\boldsymbol{e}) - K_1\|\boldsymbol{\sigma}\|_\infty^{\frac{p-1}{p}} sign(\boldsymbol{\sigma}) + \boldsymbol{V}$$

$$\dot{\boldsymbol{V}} = -K_2\|\boldsymbol{\sigma}\|_\infty^{\frac{p-2}{p}} sign(\boldsymbol{\sigma}) \quad (12)$$

with $\boldsymbol{e} = \boldsymbol{clmn}(\tilde{\boldsymbol{x}}_i) \in \mathbb{R}^N$, $\boldsymbol{V} = \boldsymbol{clmn}(\boldsymbol{v_i}) \in \mathbb{R}^N$, $\boldsymbol{F}(\boldsymbol{e}) = \boldsymbol{clmn}\left(\left(\boldsymbol{f}_i(\boldsymbol{x}) - \boldsymbol{f}_0(\boldsymbol{x})\right)\right) \in \mathbb{R}^N$, where the vector $\boldsymbol{clmn}(\boldsymbol{z}_i)$ denotes a column vector created from the sequence of vectors $\boldsymbol{z}_i$ for $i = 1, \dots, N = mn$.

Using expression (9), the sliding variable vector in (12) is defined as follows

$$\boldsymbol{\sigma} = (\boldsymbol{M} \otimes \boldsymbol{I}_N)\boldsymbol{e} \quad (13)$$

where the matrix $\boldsymbol{M}$ is as defined in assumption 3, $\boldsymbol{I}_N$ denotes the identity matrix of order N, and the symbol $\otimes$ denotes the Kronecker product.

Assumption 7: Suppose that the dynamics (5) are bounded, $\lambda_{max}(\boldsymbol{M}) > 0$, and there exists a pair of constants $l, \delta \in \mathbb{R}^+$, for which

$$\|\boldsymbol{F}(\boldsymbol{e})\|_\infty \leq \delta\|\boldsymbol{\sigma}\|_\infty^{\frac{p-1}{p}} \|(\boldsymbol{M} \otimes \boldsymbol{I}_N)\|_\infty \leq l\lambda(\boldsymbol{M})_{max} \quad (14)$$

Theorem 1: Consider that assumptions 1-4 and 6-7 hold. If the fixed undirected graph $\mathcal{G}$ is connected with at least one $a_{i0} > 0$, the distributed protocols (10) enforce the followers' dynamics (6) to satisfy the consensus condition (8) provided that the gains K_1 and K_2 are selected high enough so that

$$\frac{\lambda}{min^{\lambda(\boldsymbol{P})max}{}_{1/2(\boldsymbol{P})(\widehat{\boldsymbol{Q}})_{min}}}, \quad \boldsymbol{P} = \frac{1}{2}\begin{bmatrix} 4K_2 + K_1^2 & -K_1 \\ -K_1 & 2 \end{bmatrix} \quad (15)$$

$$\hat{Q}_{11} = K_1K_2 - K_1\left(4K_2 + K_1^2\right)l\lambda_{\max}\left(\boldsymbol{M}\right) - \left(4K_2 + K_1^2\right)l\lambda_{\max}\left(\boldsymbol{M}\right)\delta$$

$$\hat{Q}_{12} = \hat{Q}_{21} = -K_2 - 2K_2l\lambda_{\max}\left(\boldsymbol{M}\right) + \frac{K_1}{2}l\lambda_{\max}\left(\boldsymbol{M}\right)\delta$$

$$\hat{Q}_{22} = K_1l\lambda_{\max}\left(\boldsymbol{M}\right) \quad (16)$$

Proof: Consider the expression (10) in the case of $p = 2$ and a modified form of the Lyapunov function candidate proposed in [18].

$$V = 2K_2\|\boldsymbol{\sigma}\|_\infty + \frac{1}{2}\|\boldsymbol{V}\|_\infty^2 + \frac{1}{2}\left(K_1\sqrt{\|\boldsymbol{\sigma}\|_\infty} - \|\boldsymbol{V}\|_\infty\right)^2 = \frac{1}{2}\boldsymbol{\xi}^T\boldsymbol{P}\boldsymbol{\xi} \quad (17)$$

where $\|\boldsymbol{z}\|_\infty$ denotes the infinity norm of a vector $\boldsymbol{z}$ and

$$\boldsymbol{\xi} = \left[\sqrt{\|\boldsymbol{\sigma}\|_\infty} \quad \|\boldsymbol{V}\|_\infty\right]^T \quad (18)$$

The time derivative $\dot{V}$ is calculated as

$$\dot{V} = \frac{1}{2}\boldsymbol{\xi}^T\boldsymbol{P}\dot{\boldsymbol{\xi}} + \frac{1}{2}\dot{\boldsymbol{\xi}}^T\boldsymbol{P}\boldsymbol{\xi}$$
$$= \frac{1}{2}\boldsymbol{\xi}^T\boldsymbol{P}\left[\|\dot{\boldsymbol{\sigma}}\|_\infty/\left(2\sqrt{\|\boldsymbol{\sigma}\|_\infty}\right)sign(\sigma_p) \quad \left\|\dot{\boldsymbol{V}}\right\|_\infty\right]^T + \quad (19)$$
$$1/2\left[\|\dot{\boldsymbol{\sigma}}\|_\infty/\left(2\sqrt{\|\boldsymbol{\sigma}\|_\infty}\right)sign(\sigma_p) \quad \left\|\dot{\boldsymbol{V}}\right\|_\infty\right]^T\boldsymbol{P}\boldsymbol{\xi}$$

where σ_p is defined such that $\|\boldsymbol{\sigma}\|_\infty = \left|\sigma_p\right|$. With $\dot{\xi}_2 = \left\|\dot{\boldsymbol{V}}\right\|_\infty = \|-K_2 sign(\boldsymbol{\sigma})\|_\infty = K_2$, expression (20) becomes

$$V \approx \frac{-K_2}{2}\left[K_1\sqrt{\|\boldsymbol{\sigma}\|_\infty} - 2\|\boldsymbol{V}\|_\infty\right] - \|\dot{\boldsymbol{\sigma}}\|_\infty/$$
$$\left(2\sqrt{\|\boldsymbol{\sigma}\|_\infty}\right)sign(\sigma_p)\left[-(4K_2 + K_1^2)\sqrt{\|\boldsymbol{\sigma}\|_\infty} + K_1\|\boldsymbol{V}\|_\infty\right] \quad (20)$$

Using the following norm properties:

$$\|\dot{\boldsymbol{\sigma}}\|_\infty = \|\boldsymbol{M} \otimes \boldsymbol{I}_N\dot{\boldsymbol{e}}\|_\infty \leq \|\boldsymbol{M} \otimes \boldsymbol{I}_N\|_\infty\|\dot{\boldsymbol{e}}\|_\infty$$
$$\|\dot{\boldsymbol{e}}\|_\infty \leq \|\boldsymbol{F}(\boldsymbol{e})\|_\infty + K_1\sqrt{\|\boldsymbol{\sigma}\|_\infty} + \|\boldsymbol{V}\|_\infty \quad (21)$$

expression (20) can be written as

$$\dot{V} \approx \frac{-K_2}{2}\left[K_1\sqrt{\|\boldsymbol{\sigma}\|_\infty} - 2\|\boldsymbol{V}\|_\infty\right]$$
$$- 1/\left(\sqrt{\|\boldsymbol{\sigma}\|_\infty}\right)\|(\boldsymbol{M} \otimes \boldsymbol{I}_M)\|_\infty.$$
$$\left(\|\boldsymbol{F}(\boldsymbol{e})\|_\infty + K_1\sqrt{\|\boldsymbol{\sigma}\|_\infty} + \|\boldsymbol{V}\|_\infty\right).\left[-(4K_2 + K_1^2)\sqrt{\|\boldsymbol{\sigma}\|_\infty} + K_1\|\boldsymbol{V}\|_\infty\right] \quad (22)$$

In matrix form,

$$\dot{V} \approx -\frac{1}{2\sqrt{\|\boldsymbol{\sigma}\|_\infty}}\boldsymbol{\xi}^T\boldsymbol{Q}_1\boldsymbol{\xi} - \frac{\|(\boldsymbol{M}\otimes\boldsymbol{I}_N)\|_\infty}{2\sqrt{\|\boldsymbol{\sigma}\|_\infty}}\boldsymbol{\xi}^T\boldsymbol{Q}_2\boldsymbol{\xi} - \frac{\|(\boldsymbol{M}\otimes\boldsymbol{I}_N)\|_\infty}{2\sqrt{\|\boldsymbol{\sigma}\|_\infty}}\|\boldsymbol{F}(\boldsymbol{e})\|_\infty\boldsymbol{q}^T\boldsymbol{\xi} \quad (23)$$

with

$$\boldsymbol{Q}_2 = \begin{bmatrix} -K_1(4K_2 + K_1^2) & -2K_2 \\ -2K_2 & K_1 \end{bmatrix}$$

$$\boldsymbol{Q}_1 = K_2\begin{bmatrix} K_1 & -1 \\ -1 & 0 \end{bmatrix}, \boldsymbol{q}^T = \begin{bmatrix} -(4K_2 + K_1^2) & K_1 \end{bmatrix} \quad (24)$$

According to assumption 7, expression (23) reduces to

$$\dot{V} \approx -1/\left(2\sqrt{\|\boldsymbol{\sigma}\|_\infty}\right).\left(\boldsymbol{\xi}^T\boldsymbol{Q}_1\boldsymbol{\xi} + l\lambda(\boldsymbol{M})^T{}_2(\boldsymbol{M})^T{}_{3_{max_{max}}}\right) \quad (25)$$

with

$$\boldsymbol{Q}_3 = \begin{bmatrix} -(4K_2 + K_1^2) & \frac{K_1}{2} \\ \frac{K_1}{2} & 0 \end{bmatrix} \quad (26)$$

In compact form,

$$\dot{V} \approx -1/\left(2\sqrt{\|\boldsymbol{\sigma}\|_\infty}\right)\boldsymbol{\xi}^T\widehat{\boldsymbol{Q}}\boldsymbol{\xi} \quad (27)$$

$$\hat{Q}_{11} = K_1K_2 - K_1\left(4K_2 + K_1^2\right)l\lambda_{\max}\left(\boldsymbol{M}\right) - \left(4K_2 + K_1^2\right)l\lambda_{\max}\left(\boldsymbol{M}\right)\delta$$

$$\hat{Q}_{12} = \hat{Q}_{21} = -K_2 - 2K_2l\lambda_{\max}\left(\boldsymbol{M}\right) + \frac{K_1}{2}l\lambda_{\max}\left(\boldsymbol{M}\right)\delta, \ \hat{Q}_{22} = K_1l\lambda_{\max}\left(\boldsymbol{M}\right) \quad (28)$$

From the following inequalities:

$$\dot{V} \approx -1/\left(2\sqrt{\|\boldsymbol{\sigma}\|_\infty}\right)\boldsymbol{\xi}^T\widehat{\boldsymbol{Q}}\boldsymbol{\xi} \leq -1/$$
$$\left(2\sqrt{\|\boldsymbol{\sigma}\|_\infty}\right)\lambda\left(\widehat{\boldsymbol{Q}}\right)\|\boldsymbol{\xi}\|_2^2{}_{min}\sqrt{\|\boldsymbol{\sigma}\|_\infty} \leq \|\boldsymbol{\xi}\|_2 \leq \sqrt{V}/$$
$$\lambda_{min}^{1/2(\boldsymbol{P}) \quad (\boldsymbol{P})\|\xi\|_2^2(\boldsymbol{P})\|\xi\|_2^2{}_{max_{min}}} \quad (29)$$

it results that

$$\dot{V} \leq -\gamma\sqrt{V}, \gamma = \lambda_{min}^{1/2(\boldsymbol{P})(\widehat{\boldsymbol{Q}})(\boldsymbol{P})_{max_{min}}} \quad (30)$$

End of proof.

The convergence time (settling time) can be estimated from the following expression:

$$\sqrt{V} = \sqrt{V_0} - \frac{1}{2}\gamma t \quad (31)$$

Let $\sqrt{V_0} - \frac{1}{2}\gamma t^* = 0$, which gives the convergence time t^* as

$$t^* = 1/\gamma \boldsymbol{\xi}_0^T \boldsymbol{P} \boldsymbol{\xi}_0 \quad (32)$$

Lemma 2: The Lyapunov function (17) ensures the convergence of all trajectories of the consensus (11) to zero in a finite time t equal or smaller than t^*.

Lemma 3: Since the Lyapunov function (17) is continuous everywhere but not differentiable at $\|\boldsymbol{\sigma}\|_\infty = 0$ (except on the set $S = \{\|\boldsymbol{\sigma}\|_\infty, \|\boldsymbol{V}\|_\infty \in \mathbb{R}^2 \mid \|\boldsymbol{\sigma}\|_\infty = 0\}$), the solutions of the consensus (11) are understood in Filippov's sense. Hence, the function (17) is not locally Lipschitz function.

Lemma 4: In the case of a fixed directed graph topology, the results obtained in theorem 1 remain valid with substitution of the matrix $\boldsymbol{M}$ in (13) by a matrix $\boldsymbol{N}$ such that

$$\boldsymbol{\sigma} = (\boldsymbol{N} \otimes \boldsymbol{I}_N)\boldsymbol{e}, \ \mathbf{NM} + \mathbf{M}^T \boldsymbol{N} = I_N \quad (33)$$

Remark. The gains K_1 and K_2 in protocols (10) can be tuned along each motion direction to get enough smooth control input.

C. Perturbed Dynamics

Consider the following perturbed consensus dynamics model

$$\dot{\tilde{\boldsymbol{x}}}_i = \boldsymbol{f}_i(\boldsymbol{x}_i) - \boldsymbol{f}_0(\boldsymbol{x}_0) + \boldsymbol{u}_i(\tilde{\boldsymbol{x}}_i) + \boldsymbol{d}_i(\tilde{\boldsymbol{x}}_i) \quad (34)$$

Assumption 8. The disturbances $\boldsymbol{d}_i(t)$ are bounded disturbances that satisfy the following conditions

$$\boldsymbol{d}_i(\boldsymbol{x}_i, t) = \boldsymbol{d}_i^s(\boldsymbol{x}_i, t) + \boldsymbol{d}_i^u(\boldsymbol{x}_i, t), \ \lim_{t\to\infty} \boldsymbol{d}_i(\boldsymbol{x}_i, t) = \boldsymbol{\zeta}_i \quad (35)$$

where $\boldsymbol{d}_i^s(\boldsymbol{x}_i, t)$ and $\boldsymbol{d}_i^u(\boldsymbol{x}_i, t)$ denote the structured and unstructured parts of the matched disturbances $\boldsymbol{d}_i$ and $\boldsymbol{\zeta}_i$ are unknown constant vectors.

Assumption 9. The unstructured disturbances $\boldsymbol{d}_i^u(\boldsymbol{x}_i, t)$ can be considered as zero-mean Gaussian noises while the structured disturbances $\boldsymbol{d}_i^s(\boldsymbol{x}_i, t)$ are expressed using regressor notation [18]

$$d_{i,k}^s(\boldsymbol{x}_i, t) = \boldsymbol{\theta}_i^T \boldsymbol{\varphi}_i(\boldsymbol{x}_i) \quad k = 1,2,\ldots,m \quad (36)$$

where $\boldsymbol{\theta}_i \in \mathbb{R}^p$ is an uncertain parameter vector and $\boldsymbol{\varphi}_i: \mathbb{R}^m \to \mathbb{R}^p$ is a known nonlinear base function. In the presence of structured disturbances (35), the consensus dynamics (12) are rewritten as

$$\dot{\boldsymbol{e}} = \boldsymbol{F}(\boldsymbol{e}) - K_1 \|\boldsymbol{\sigma}\|_\infty^{\frac{p-1}{p}} sign(\boldsymbol{\sigma}) + \boldsymbol{V} - \boldsymbol{\Theta}^T \boldsymbol{\Phi}(\boldsymbol{x})$$
$$\dot{\boldsymbol{V}} = -K_2 \|\boldsymbol{\sigma}\|_\infty^{\frac{p-2}{p}} sign(\boldsymbol{\sigma}) \quad (37)$$

where

$$\boldsymbol{\Theta} = [\theta_1^T, \theta_2^T, \ldots, \theta_n^T]^T \in R^N, \quad \Phi = [\varphi_1^T, \varphi_2^T, \ldots, \varphi_n^T]^T \in R^N \quad (38)$$

Theorem 2: Consider that assumptions 4 and 5 hold. If the graph $\mathcal{G}$ is connected with at least one $a_{i0} > 0$, the following agents' controllers and disturbance observers ensure that the consensus condition (8) is robustly achieved in finite time despite external disturbances.

Controllers:

$$u_{i,k} = -K_1 \|\boldsymbol{\sigma}_k(\boldsymbol{x}_i)\|_\infty^{\frac{p-1}{p}} sign\left(\sigma_{i,k}(\boldsymbol{x}_i)\right) + v_{i,k} - \boldsymbol{\theta}_i^T \boldsymbol{\varphi}_i(\tilde{\boldsymbol{x}}_i)$$
$$\dot{v}_{i,k} = -K_2 \|\boldsymbol{\sigma}_k(\boldsymbol{x}_i)\|_\infty^{\frac{p-2}{p}} sign\left(\sigma_{i,k}(\boldsymbol{x}_i)\right) \quad (38)$$

Observers:

$$\dot{\hat{\boldsymbol{\theta}}}_i = \boldsymbol{\Gamma}_i \boldsymbol{\Psi}_i(\boldsymbol{\sigma}_i) \boldsymbol{\varphi}_i(\tilde{\boldsymbol{x}}_i) \quad (39)$$

where $\boldsymbol{\Gamma}_i = \boldsymbol{diag}(\rho_{1,1}, \rho_{1,2}, \ldots, \rho_{1,m}) \in \mathbb{R}^{m\times m}$ and $\boldsymbol{\Psi}_i(\boldsymbol{\sigma}_i) = \boldsymbol{diag}(sign(\sigma_{i,j})) \in \mathbb{R}^{m\times m}$.

Proof: Consider the following Lyapunov function

$$V_{ext} = V_{nom} + \frac{1}{2} \tilde{\boldsymbol{\Theta}}^T \boldsymbol{\Gamma}^{-1} \tilde{\boldsymbol{\Theta}} \quad (40)$$

where V_{nom} is given by expression (17), $\tilde{\boldsymbol{\Theta}} = (\hat{\boldsymbol{\Theta}} - \boldsymbol{\Theta}) \in \mathbb{R}^N$ is a parameter error vector, $\hat{\boldsymbol{\Theta}}$ is the estimate of the unknown parameter vector $\boldsymbol{\Theta}$, and $\boldsymbol{\Gamma} = \boldsymbol{diag}(\rho_{1,1}, \ldots, \rho_{1,m}, \ldots \rho_{n,1}, \ldots, \rho_{n,m}) \in \mathbb{R}^{N\times N}$ with $\rho_{i,j}$ being adaptive gain coefficient for the agent 'i' along motion direction 'j'. To actively estimate and reject external disturbances in each agent's motion direction and robustly achieve consensus tracking (8), the following adaptive law is proposed.

$$\dot{\hat{\boldsymbol{\Theta}}} = \boldsymbol{\Gamma} \boldsymbol{\Psi}(\boldsymbol{\sigma}) \Phi(\boldsymbol{x}) \quad (41)$$

where $\boldsymbol{\Psi}(\boldsymbol{\sigma}) = \boldsymbol{diag}(sign(\sigma_{i,j})) \in \mathbb{R}^{N\times N}$. Since $\boldsymbol{\Theta}$ is unknown, the time-derivative of (44) is obtained as

$$\dot{V}_{ext} = \dot{V}_{nom} + \tilde{\boldsymbol{\Theta}}^T \boldsymbol{\Gamma}^{-1} \dot{\hat{\boldsymbol{\Theta}}} \quad (42)$$

With (30), the extended Lyapunov function may be bounded as

$$\dot{V}_{ext} \le -\left[\gamma\sqrt{V_{nom}} + \tilde{\boldsymbol{\Theta}}^T \boldsymbol{\Gamma}^{-1}\left(\dot{\hat{\boldsymbol{\Theta}}} - \boldsymbol{\Gamma}\boldsymbol{\Psi}(\boldsymbol{\sigma})\Phi(\boldsymbol{x})\right)\right] \quad (43)$$

end of the proof.

IV. Consensus-tracking for Second-order Dynamics

A. Problem Statement

This section addresses the design of distributed consensus tracking protocols for nonlinear second-order MAS to achieve robust high-accuracy position and velocity consensus tracking. Consider a MAS composed of a virtual leader '0' and n identical followers with nonlinear uncertain second-order dynamics subject to unknown but bounded external disturbances. The leader's and followers' dynamics are, respectively

$$\dot{\boldsymbol{x}}_0 = \boldsymbol{v}_0 \dot{\boldsymbol{v}}_0 = \boldsymbol{f}_0(\boldsymbol{x}_0) + \boldsymbol{G}_0(\boldsymbol{x}_0)\boldsymbol{u}_0(\boldsymbol{x}_0) \quad (44)$$

$$\dot{\boldsymbol{x}}_i = \boldsymbol{v}_i \dot{\boldsymbol{v}}_i = \boldsymbol{f}_i(\boldsymbol{x}_i) + \boldsymbol{G}_i(\boldsymbol{x}_i)[\boldsymbol{u}_i(\boldsymbol{x}_i) + \boldsymbol{d}_i(\boldsymbol{x}_i, t)] \quad (45)$$

where $\boldsymbol{x}_0 \in \mathbb{R}^m$ and $\boldsymbol{v}_0 \in \mathbb{R}^m$ are the leader's state and velocity vectors, respectively; $\boldsymbol{x}_i \in \mathbb{R}^m$, $\boldsymbol{v}_i \in \mathbb{R}^m$, $\boldsymbol{u}_i \in \mathbb{R}^m$, and $\boldsymbol{d}_i \in \mathbb{R}^m$ are the i^{th} follower's state, velocity, control input, and disturbance vectors, respectively; $\boldsymbol{f}_0 \in \mathbb{R}^m$, $\boldsymbol{f}_i \in \mathbb{R}^m$, $\boldsymbol{d}_i \in \mathbb{R}^m$, $\boldsymbol{G}_0 \in \mathbb{R}^{m\times m}$ and $\boldsymbol{G}_i \in \mathbb{R}^{m\times m}$ are continuous uncertain functions. Disturbances $\boldsymbol{d}_i$ obey the conditions in assumptions 5, 8 and 9.

Assumption 10: The control matrices $\boldsymbol{G}_0$ and $\boldsymbol{G}_i$ are defined such that $\boldsymbol{G}_0 = \boldsymbol{diag}(1/\rho_{01}^2, .., 1/\rho_{0m}^2)$ and $\boldsymbol{G}_i = \boldsymbol{diag}(1/\rho_1^2, .., 1/\rho_m^2)$ where ρ_j denotes the control constraints along the $'j'$ motion direction.

The objective of second-order distributed consensus tracking is to design protocols $\boldsymbol{u}_i$ for dynamics (49) such that the following consensus agreement is achieved simultaneously by all the followers' dynamics and maintained for further time:

$$\lim_{t\to T}\|\boldsymbol{x}_i(t) - \boldsymbol{x}_0(t)\| = 0, \lim_{t\to T}\|\boldsymbol{v}_i(t) - \boldsymbol{v}_0(t)\| = 0 \quad \forall i = 1,2,\dots,n \tag{46}$$

To apply STSM control to the second-order distributed consensus tracking problem, the sliding variables are defined, for $i = 1,..,n$ $k = 1,..,m$, as follows:

$$\sigma_{i,k}(\boldsymbol{x}_i) = \sum_{j=0}^{n} a_{ij}\left[x_{i,k} - x_{j,k}\right] + c\sum_{j=0}^{n} a_{ij}\left[v_{i,k} - v_{j,k}\right] \tag{47}$$

where $c \in \mathbb{R}^+$.

B. Second-order Distributed Consensus Tracking

To address the problem of second-order distributed consensus tracking in its general form, the leader's dynamics are considered nonlinear dynamics with time-varying velocities. For n agents and m motion directions, the sliding manifold (48) and the consensus dynamics (44)-(45) are written, in matrix form, as follows:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_x + c\boldsymbol{\sigma}_v \tag{48}$$

$$\begin{gathered}\dot{\boldsymbol{e}}_x = \boldsymbol{e}_v\\ \dot{\boldsymbol{e}}_v = \boldsymbol{F}(\boldsymbol{e}_v) + (\boldsymbol{M}\otimes\boldsymbol{I}_N)\big(\boldsymbol{G}(\boldsymbol{e}_v)\boldsymbol{U} - \boldsymbol{U}_0(\boldsymbol{x}_0)\big)\end{gathered} \tag{49}$$

where $\boldsymbol{e}_x = [\tilde{\boldsymbol{x}}_1^T, \dots, \tilde{\boldsymbol{x}}_n^T]^T \in \mathbb{R}^N$, $\boldsymbol{e}_v = [\tilde{\boldsymbol{v}}_1^T, \dots, \tilde{\boldsymbol{v}}_n^T]^T \in \mathbb{R}^N$, $\boldsymbol{\sigma} = [\boldsymbol{\sigma}_1^T, \dots, \boldsymbol{\sigma}_n^T]^T \in \mathbb{R}^{2N}$, and c is a positive constant. The vectors $\tilde{\boldsymbol{x}}_i$ are defined as in the previous section $\tilde{\boldsymbol{x}}_i = \boldsymbol{x}_i - \boldsymbol{x}_0 \in \mathbb{R}^m$, $\tilde{\boldsymbol{v}}_i = \boldsymbol{v}_i - \boldsymbol{v}_0 \in \mathbb{R}^m$, and $\boldsymbol{\sigma}_i = \left[\sigma_{i,1}, .., \sigma_{i,m}\right]^T \in \mathbb{R}^{2m}$; $\boldsymbol{U} = [\boldsymbol{u}_1^T, \dots, \boldsymbol{u}_n^T]^T \in \mathbb{R}^N$, $\boldsymbol{G}(\boldsymbol{e}_v) = [\boldsymbol{G}_1, \dots, \boldsymbol{G}_n]^T \in \mathbb{R}^{N\times m}$, and $\boldsymbol{U}_0 = \boldsymbol{rep}((\boldsymbol{G}_0\boldsymbol{u}_0)^T, N)^T \in \mathbb{R}^N$ with $(\boldsymbol{rep}(\boldsymbol{z}), n)$ denotes a vector formed by n replications of the vector $\boldsymbol{z}$.

Assumption 11. The following upper limit bounds the leader's control inputs

$$\|\boldsymbol{G}_0(\boldsymbol{x}_0)\boldsymbol{u}_0\|_\infty \le v_{0,max} \tag{50}$$

where $v_{0,max} \in \mathbb{R}^+$ is a control constraint.

Assumption 12: Suppose that dynamics (45) are bounded, $\lambda_{max}(\boldsymbol{M}) > 0$, and there exist some constants $l_M, l_G, \delta_v \in \mathbb{R}^+$, for which

$$\begin{gathered}\|\boldsymbol{F}(\boldsymbol{e}_v)\|_\infty \le \delta_v\|\boldsymbol{\sigma}\|_\infty^{\alpha_1}\\ \|\boldsymbol{G}(\boldsymbol{e}_v)\|_\infty \le l_G\lambda(\boldsymbol{G})_{max}\\ \|(\boldsymbol{M}\otimes\boldsymbol{I}_N)\|_\infty \le l_M\lambda(\boldsymbol{M})_{max}\end{gathered} \tag{51}$$

Theorem 3: Suppose assumptions 1-4 and 10-12 hold. The following STSM protocol enforces the MAS (48)-(49) to satisfy the consensus condition (45) in finite time despite uncertainties and/or disturbances.

$$\begin{gathered}\boldsymbol{U} = -K_1 vect\big(|\sigma_k|^{\alpha_1} sign(\sigma_k)\big) + \boldsymbol{V} - \boldsymbol{\Theta}^T\boldsymbol{\Phi}(\boldsymbol{x})\\ \dot{\boldsymbol{V}} = -K_2 vect\big(|\sigma_k|^{\alpha_2} sign(\sigma_k)\big) \quad k = 1,..N\end{gathered} \tag{52}$$

with $\alpha_2 = 2\alpha_1/(1+\alpha_1)$, $\boldsymbol{V} = [\boldsymbol{V}_1^T, \dots, \boldsymbol{V}_n^T]^T \in \mathbb{R}^N$, $\boldsymbol{V}_i \in \mathbb{R}^m$.

Proof: Consider the case of $\alpha_1 = 1/2$ in expression (52) and the nominal form of the consensus model (44)-(45) and select the following Lyapunov function:

$$V_{nom}(\boldsymbol{\xi}) = K_2\int_0^{\|\sigma\|_\infty}\|\boldsymbol{z}\|_\infty^{\alpha_2}dz + \frac{1}{2}\|\boldsymbol{V}\|_\infty^2 \tag{53}$$

$$\boldsymbol{\xi} = [\|\boldsymbol{\sigma}\|_\infty \quad \|\boldsymbol{V}\|_\infty]^T \tag{54}$$

The time-derivative $\dot{V}_{nom}$ can be given as

$$\dot{V}_{nom} = \partial V/\partial\boldsymbol{\xi}.\dot{\boldsymbol{\xi}} = \langle K_2\|\boldsymbol{\sigma}\|_\infty^{\alpha_2} \quad \|\boldsymbol{V}\|_\infty\rangle\big[\|\dot{\boldsymbol{\sigma}}\|_\infty \quad \|\dot{\boldsymbol{V}}\|_\infty\big]^T \tag{55}$$

Assuming that

$$\|\dot{\boldsymbol{\sigma}}_x\|_\infty = -c\|\dot{\boldsymbol{\sigma}}_v\|_\infty \tag{56}$$

It results from expressions (49), (51) and (55) that

$$\begin{gathered}\dot{V}_{nom} \le \langle K_2\|\boldsymbol{\sigma}\|_\infty^{\alpha_2} \quad \|\boldsymbol{V}\|_\infty\rangle.\\ [(1-c)\|\boldsymbol{M}\otimes\boldsymbol{I}_N\|\infty\big(\|\boldsymbol{F}(\boldsymbol{e}_v)\|_\infty + \|\boldsymbol{G}(\boldsymbol{e}_v)\|_\infty\big(K_1\|\boldsymbol{\sigma}\|_\infty^{\alpha_1} + \|\boldsymbol{V}\|_\infty\big) + \|\boldsymbol{U}_0\|_\infty\big) \quad K_2\|\boldsymbol{\sigma}\|_\infty^{\alpha_2}]^T\end{gathered} \tag{57}$$

with

$$c = 1 + \lambda v(\lambda v max_{max})max_{max} \tag{58}$$

and

$$\dot{V}_{nom} \le -\frac{\|\boldsymbol{M}\otimes\boldsymbol{I}_N\|_\infty K_2}{\lambda_{max}\big(\boldsymbol{G}(\boldsymbol{e}_v)\big)}\|\boldsymbol{\sigma}\|_\infty^{\alpha_2}.$$

$$\big(\|\boldsymbol{F}(\boldsymbol{e}_v)\|_\infty + K_1\|\boldsymbol{G}(\boldsymbol{e}_v)\|_\infty\|\boldsymbol{\sigma}\|_\infty^{\alpha_1} + \|\boldsymbol{G}(\boldsymbol{e}_v)\|_\infty\|\boldsymbol{U}_0\|_\infty\big) \tag{59}$$

Using the bounds (55) and (56), it results that

$$\dot{V}_{nom} \le -\frac{\lambda_{max}(\boldsymbol{M})l_M K_2}{\lambda_{max}\big(\boldsymbol{G}(\boldsymbol{e}_v)\big)}.$$

$$\begin{pmatrix}\delta_v\|\boldsymbol{\sigma}\|_\infty^{(\alpha1+\alpha_2)} + K_1 l_G\lambda_{max}(\boldsymbol{G}(\boldsymbol{e}_v))\|\boldsymbol{\sigma}\|_\infty^{(\alpha1+\alpha_2)}\\ +l_G\lambda_{max}(\boldsymbol{G}(\boldsymbol{e}_v))v\|\boldsymbol{\sigma}\|_\infty^{\alpha_2}{}_{0,max}\end{pmatrix} \tag{60}$$

end of the proof.

Lemma 5: Since V ̇_nom is not strictly negative because V ̇_nom=0 for ‖σ‖_∞=0, the asymptotic stability of the consensus tracking is guaranteed by the Krasovskii-LaSalle's invariance principle.

Proof of Lemma 5: Let $S = \big\{(\|\boldsymbol{\sigma}\|_\infty, \|\boldsymbol{V}\|_\infty) \in \mathbb{R}^{\boldsymbol{2}}: \dot{V}_{nom} = 0\big\}$, the asymptotic stability of the consensus tracking is guaranteed only if $S = \{(0,0)\}$. For $\lambda_{max}(\boldsymbol{M}) > 0$ and $\lambda_{max}(\boldsymbol{G}) > 0$, equation (64) has $\|\boldsymbol{\sigma}_v\|_\infty = 0$ as the only solution for $\dot{V}_{nom} = 0$. From the dynamics (49) and (52) the only remaining solution is $\|\boldsymbol{V}\|_\infty = 0$.

Lemma 6: In the case of structured disturbances, the asymptotic convergence of the extended Lyapunov function

$$V_{ext} = K_2 \int_0^{\|\boldsymbol{\sigma}\|_\infty} \|\boldsymbol{z}\|_\infty^{1/3} dz + \frac{1}{2}\|\boldsymbol{V}\|_\infty^2 + \widetilde{\boldsymbol{\Theta}}^T \boldsymbol{\Gamma}^{-1} \dot{\widehat{\boldsymbol{\Theta}}} \quad (61)$$

is guaranteed by the same conditions as in (63) and the observers (39) can be used to estimate the structured disturbances.

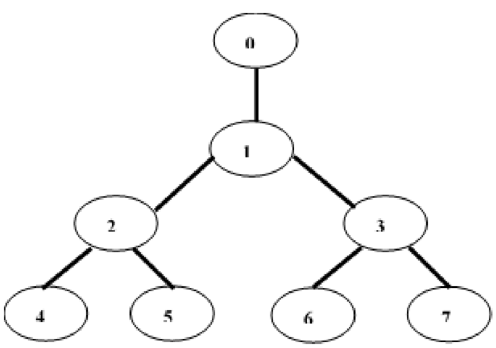
V. Simulation

The proposed consensus protocols and observers' effectiveness are evaluated in this section. Both first-order and second-order control algorithms are run using the Matlab simulation environment with a sampling time $\Delta t = 0.0001\ sec$.

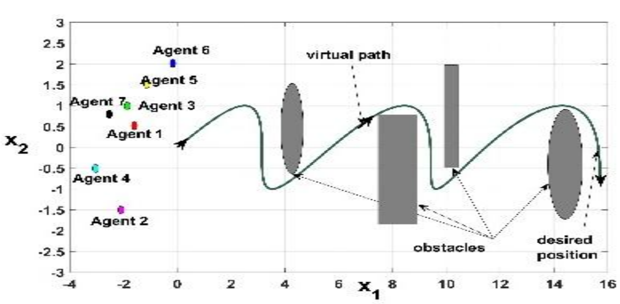
A. First-order Planar Consensus without Disturbances

Consider a network of seven agents indexed by '1' to '7', respectively, to follow a virtual leader indexed by '0'performing the virtual graph under the undirected communication topology shown in Fig. 1(a). Starting from a given initial condition, the agents must follow a common path $(x_{0,1} = t + \sin(t), x_{0,2} = \sin(\pi t/3))$ to reach a desired position while avoiding obstacles as shown in Fig. 1(b). The dynamics of the leader are given by $\dot{\boldsymbol{x}}_0 = sin(\boldsymbol{x}_0(t))$. The conventional distributed consensus controllers (62) are applied to agents $i = 1,..,n$, with $\alpha = 100$, and $\beta = 25$. The results of the consensus tracking are shown in Fig. 2 to 4.

$$u_{i,k}(\boldsymbol{x}_i) = -\alpha \sum_{j=0}^{n} a_{ij}(x_{i,k} - x_{j,k}) - \beta sign\left(\sum_{j=0}^{n} a_{ij}(x_{i,k} - x_{j,k})\right) \quad (62)$$

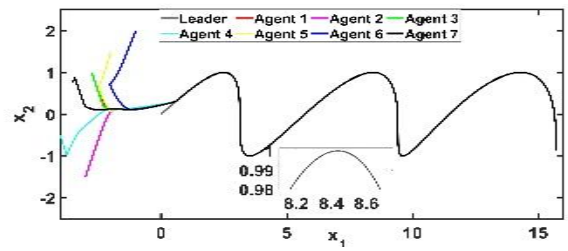


(a)

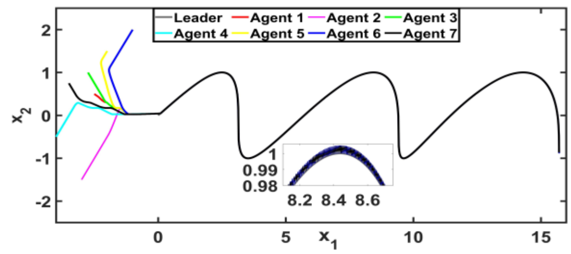


(b)

Fig. 1. Distributed consensus of seven agents: (a) Communication graph, (b) Virtual tracking path.

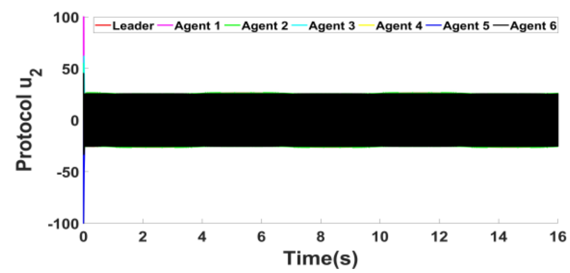


(a)

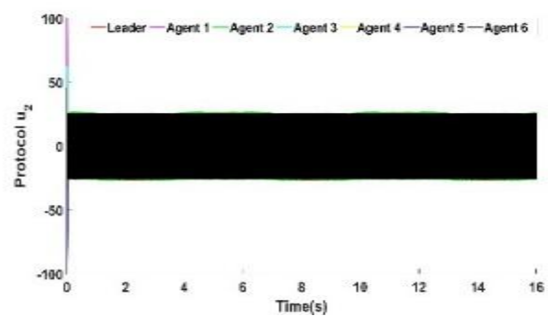


(b)

Fig. 2. Trajectories: (a) Unperturbed STSM-based consensus (10), (b) FOSM-based consensus (62).



(a)



(b)

Fig. 3. Consensus protocols using FOSM-based consensus (62): (a) Control effort u_{1i}, (b) Control effort u_{2i}.

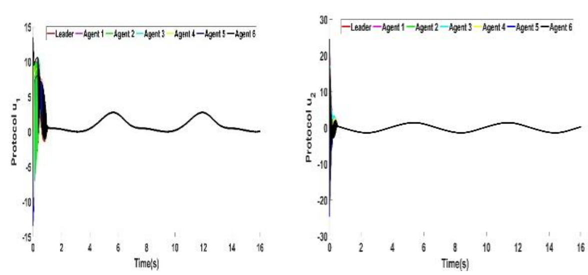


Fig. 4. Consensus protocols using STSM-based consensus (10): (a) Control effort u_{1i}, (b) Control effort u_{2i}.

B. First-order Consensus Tracking with Structured Disturbances

Consider a network of five agents indexed by '1' to '5', respectively and follow a virtual leader indexed by '0' under the communication topology shown in Fig. 5.

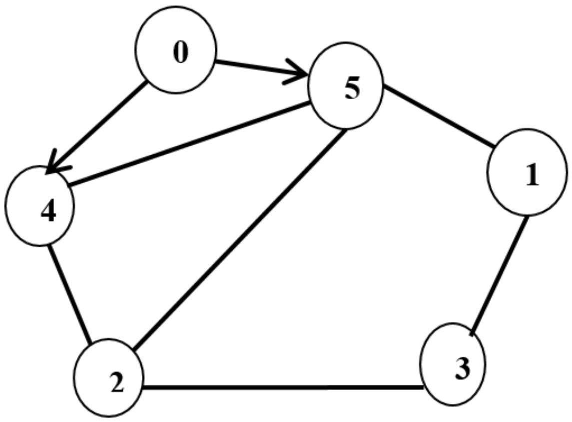


Fig. 5. The communication graph for a network of seven agents.

In this scenario, the five agents must follow a common path with the presence of structured disturbances associated with each agent's state as defined in assumption 9 with an arbitrarily selected parameter vector $\boldsymbol{\theta_i}$

$$\boldsymbol{\theta} = \begin{bmatrix} 2 & -5 & 4 & 5 & 3.5 \\ 5 & 3 & -4 & 3.6 & 2 \end{bmatrix}^T \quad (63)$$

and state-dependent base functions $\boldsymbol{\varphi_i}$

$$\boldsymbol{\varphi}_i(\boldsymbol{x}_i) = [sin(2x_{i,1}) \quad sin(2x_{i,2})]^T \quad (64)$$

The STSM-based distributed consensus protocols (9) is applied with $K_1 = 15$ and $K_2 = 30$. The disturbance observer is applied with

$$\boldsymbol{\rho} = \begin{bmatrix} 16 & 576 & 13.5 & 27 & -20 \\ 55 & 24 & 19.5 & 7.5 & 3.5 \end{bmatrix}^T \quad (65)$$

The consensus tracking, and an example for disturbance estimation and parameters updating are shown in Fig. 6 and Fig. 7.

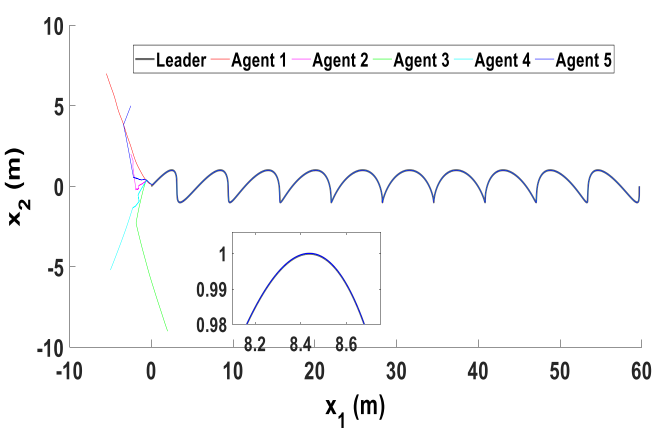


Fig. 6. Consensus tracking among the 5 agents.

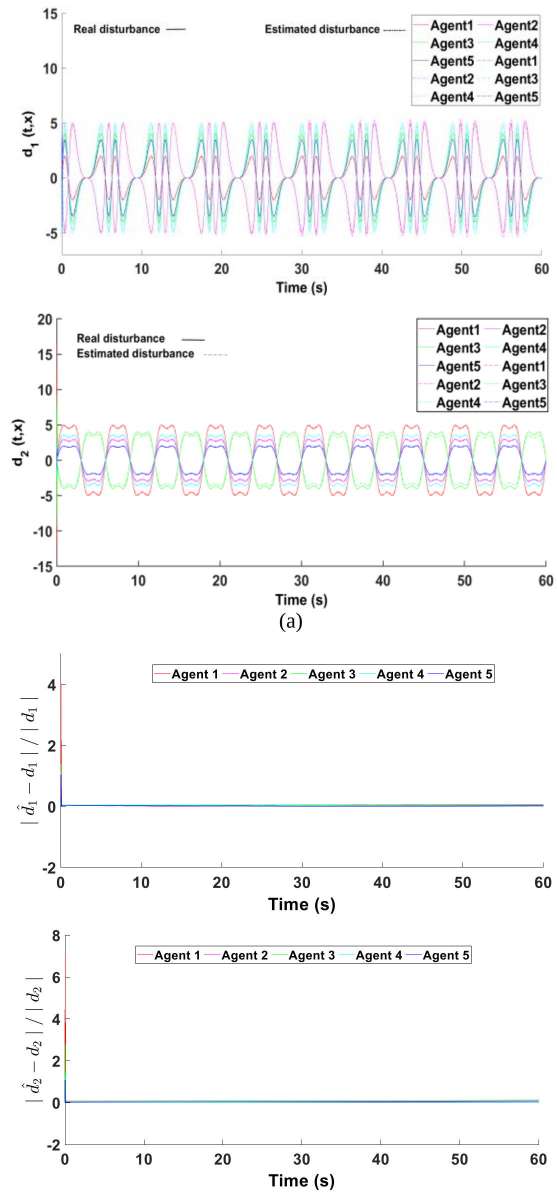
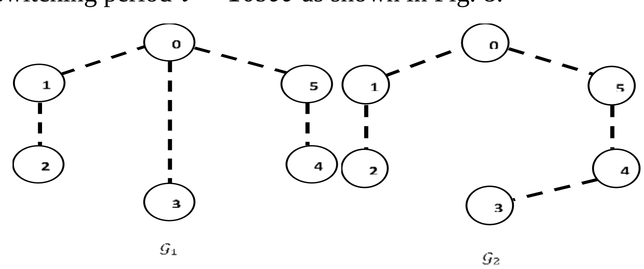


Fig. 7. Disturbance estimation for agents 1 and 2 using proposed observer: (a) Estimations (b) Estimator errors.

C. Second-order Consensus Tracking with Structured Disturbances

In this scenario, the performance and robustness of the proposed STSM-based protocol for second-order systems are simulated using a switched topology $\{\bar{\mathcal{G}}_1, \bar{\mathcal{G}}_2, \bar{\mathcal{G}}_3, \bar{\mathcal{G}}_4\}$ with switching period $\tau = 10sec$ as shown in Fig. 8.



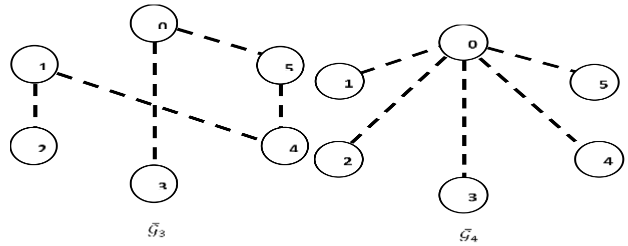
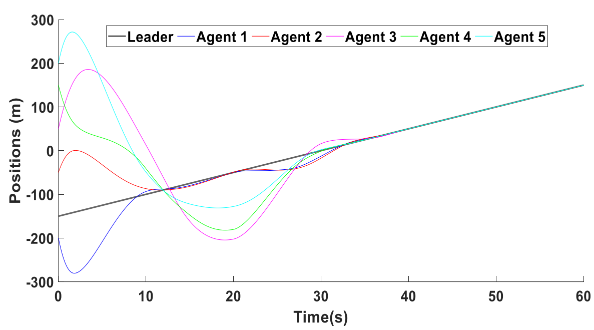


Fig. 8. Fixed-time switching topology.

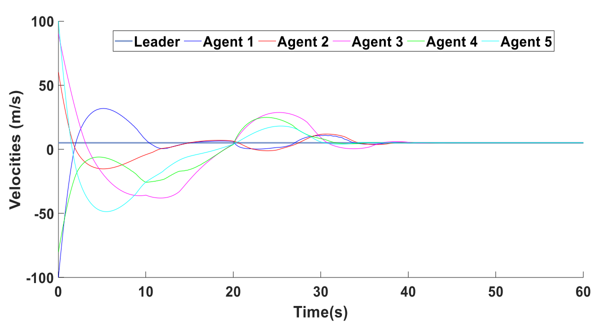
For the disturbances, an agent's state dependent component is added to the time-varying disturbances with $\theta = [1\ \ 0.5\ \ 0.6\ \ 0.8\ \ 0.2]^T$ and φ_i functions given by (66)

$$\begin{cases} \varphi_1(t, x_1) = cos(0.1t)\, sin(x_1) \\ \varphi_2(t, x_2) = sin\left(0.5\, t + \frac{\pi}{4}\right) sin(x_2) \\ \varphi_3(t, x_3) = cos(3t)\, sin(x_3) \\ \varphi_4(t, x_4)\, sin\left(2\, t + \frac{\pi}{3}\right) sin(x_4) \\ \varphi_5(t, x_5) = \begin{cases} \left((sin(\omega_1 t) - 1)\, sin(x_5)\ \text{ for t<30 sec}\right) \\ \left((sin(\omega_1 t) + 1)\, sin(x_5)\ \text{for t≥30 sec}\right) \\ \omega_1 = 2\pi\left(\frac{5.9t}{60} + 0.1\right),\ \ \omega_2 = 2\pi\left(-\frac{5.9t}{60} + 6\right) \end{cases} \end{cases} \tag{66}$$

Accurate robust finite-time consensus tracking is achieved using the proposed STSM-based protocol as shown in Fig. 9. The simulation was run with $\alpha_1 = 1/3$, $\alpha_2 = 1/2$, $c = 5$, $K_1 = 1.5$, $K_2 = 1.9$ and $\boldsymbol{\rho} = diag(10^{-3}[-20.25\ \ -4.25\ \ 9.75\ \ 19\ \ 23])$.



(a)



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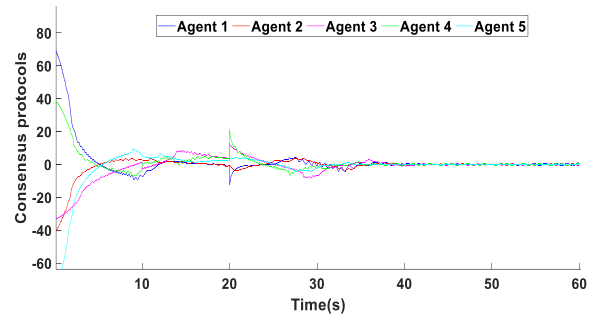
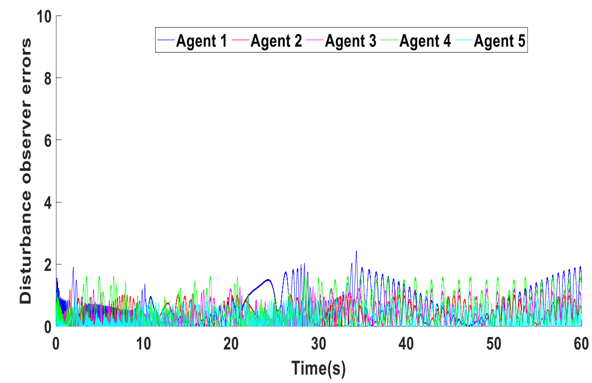


(c)



(d)

Fig. 9. Results with consensus protocol (57) (a) Trajectories, (b) Velocities, (c) Protocols (d) Disturbance estimation error.

VI. Conclusion

This paper introduced a novel finite-time synchronization framework for multi-agent systems (MAS) operating under switching communication topologies, addressing scenarios with and without direct velocity measurements. By integrating graph-theoretic principles, local finite-time convergence theory for homogeneous systems, and the non-smooth LaSalle's invariance principle, we developed a distributed control strategy ensuring precise synchronization of agents' states and velocities. The proposed control laws exhibit inherent robustness to topology variations, communication constraints, and dynamic agent interactions, making them suitable for real-world applications, including satellite formation flying, autonomous robotic networks, and cooperative unmanned aerial vehicles (UAVs).

To further enhance robustness and reduce communication overhead, we introduced a finite-time high-order sliding-mode observer, enabling agents to accurately estimate relative velocity states without direct measurements. This observer-based strategy mitigates reliance on continuous inter-agent communication, ensuring high-precision synchronization even under sensor limitations, intermittent connectivity, and external disturbances. The developed framework is inherently scalable, allowing seamless integration into large-scale distributed systems where centralized coordination is impractical or infeasible.

The results presented in this study establish a resilient and computationally efficient control paradigm for distributed synchronization in MAS, providing a strong foundation for future advancements in autonomous and cooperative multi-agent technologies. Future work will address key challenges in inter-agent communication, such as signal interference, transmission delays, and adaptive information-sharing protocols, to further enhance the real-time performance and robustness of distributed synchronization mechanisms in increasingly complex operational environments. The extension of this framework to heterogeneous agent networks, cooperative task execution, and event-triggered control will be explored to support the next generation of intelligent and autonomous multi-agent systems. Moreover, the present work could be extended beyond bounded perturbation assumptions by exploring adaptive learning-based control, stochastic models, and event-triggered MPC for real-time disturbance adaptation. Additionally, higher-order sliding mode and hybrid multi-agent reinforcement learning (MARL) approaches will be investigated to enhance robustness in highly uncertain environments. These advancements will improve the applicability of the proposed framework to real-world multi-agent systems.

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