Towards Hybrid Meta-Heuristic Analysis for the Optimization of Fundamental Performance in Robotic Systems

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***Abstract*—This paper examines the concept of implementing a hybrid optimization approach through combining analytical and meta-heuristic approaches to improve the performance of practical engineering systems. Designed in support of artificial intelligence strategy, the proposed approach ensures high stability and efficiency under actuators saturation constraint. This is a well-known and sensitive problem in robotics and control. Specifically, this paper deals with the problem of computing the stability region for controlled systems. While addressing this issue, research approaches take into consideration the fact that actuator saturation may occur. It is imperative to maintain this propriety and ensure the reliability of design control systems, particularly those developed to control robot actuators. Models of the studied systems are based on differential algebraic representations and polytypic regions in state space. The developed technique combines LMI with an improved meta-heuristic based optimization approach that fast searches and enlarge domains of attraction for robot actuators. The direct Lyapunov theory is used to analyze and validate stability key performance. A numerical example study has been conducted to validate the proposed approach's efficacy and efficiency. A comparative benchmarking study has been carried out to highlight the main concepts and results of this study.**

Keywords*—Domain of Attraction (DA); Differential Algebraic Representation (DAR); meta-heuristic approach; actuators saturation

I. Introduction

A. Motivation

In robotic systems, actuators saturation is a problem that requires careful consideration. Under faulty conditions, and/or model uncertainties, robots may be more likely to experience this problem, and its solution becomes more difficult. Among reliable robots examined are general architectures, spatial robots, and robots with parallel or serial architectures. As part of the control techniques, a model reference process is implemented as well as an estimated torque approach in the feed-forward approach [1].

Furthermore, the feedback process involves conventional controllers. In the context of stability assurance, these methods rescue the robot from unstable dynamics by considering actuator saturation during the design phase [2]. Previous studies indicate that the time regulation method coupled with basin of attraction enlargement techniques are suitable for robots control methods. Besides addressing totally failed actuator joints, these methods also provide solutions for partial defects of various actuator joints [3].

For a class of nonholonomic mobile robots, a saturated trajectory tracking control design is presented in study [1]. A bounded dynamic continuous feedback controller is developed to ensure finite-time kinematic convergence. Saturation constraints related to attraction domains as well as errors associated with tracking initial values are considered. In control systems theory, attraction domains provide a useful means of analyzing the consequences of system actuator input saturation [3]. Sets such as these identify the system initial conditions under which the control technique results in attraction to stable equilibrium points [3]. Describes these sets in the context of an exponentially unstable open-loop plant. A single actuator controls such a model that is characterized by actuator saturation and modeling time delay characteristics. Consequently, approaches to estimating attraction domains are relevant for robot control design theory since these approaches provide sufficient, and necessary conditions for attractivity.

Equally significant, these approaches allow the identification of operational stability sets in which system actuators can perform in a non-saturated state.

B. Fundamental Context

In general, in the real world, all physical control systems are inherently nonlinear: so, it is difficult to develop an analytical technique that can be applied to any nonlinear system [4]. In almost all physical control, the presence of an input saturation has been identified practically and this occurrence of saturation could result in nonlinear phenomena [4], [5], [10]. This has garnered a lot of interest to conduct numerous studies focusing on the modeling as well as assessment of its impact on the global stability and dynamical performance pertaining to closed-loop controlled systems. The results of most of these studies have accounted for the restricted class of linear open-loop systems. In such a particular case, even when the closed-loop controlled system is deemed to be linear locally , the nonlinearity could result in degradation of their performance or lead to instability. Thus, numerous analysis techniques were regarded to assess the DA pertaining to linear models subject to a control input saturation constraint. Many works can describe the saturation of inputs in different representations for the sake of facilitating stability analysis. However, a good approach would be employing the generalized sector condition pertaining to dead zone nonlinearities. Furthermore, a key part of stability analysis is the representation of the system. Dynamic nonlinear systems can be represented in a variety of ways in the literature. As a consequence, commonly used the Differential Algebraic Representations (DAR) modeling indicates that the system offers results which are conservative than the Linear Fractional Representation(LFR), as well as the Linear Parameter Varying (LPV) forms thereof. A few notable representation systems are the LPV [6], the LFR [7], [8] and the DAR. The introduction of free multipliers to the studied system may be an effective approach to decreasing this late latent conservativeness. In this way, there would be less reliance on selecting control system matrices, as suggested by [6], taking advantage of different approaches. With regards to the framework pertaining to this problem, Coutinho, Trofino and co-workers [2], [9], [10], [11], [12], [56] put forward the Differential Algebraic Representations (DAR) to enable stability analysis, deal with control synthesis problems and employ to achieve tractable stability condition as linear matrix inequalities (LMI)[10]. The authors in [2] showed that DAR results in less conservative estimates pertaining to the region of attraction. DA signifies those initial conditions in which the system state converges with that of the equilibrium in an asymptotic manner. Numerous analysis techniques have been put forward for the estimation of the DA pertaining to linear models subject to input actuators saturation constraints. Different methods have been put forward to enable calculation of inner estimates for instance, approaches such as the La Salle method [13], Zubov method [14] and the trajectory reversing. In most situations, the DA estimation problem has been segregated to non-convex or convex optimisation problems for simplification [15], [16]. This has been dealt with by employing optimisation techniques such as SOS [17], [18], intelligent optimisation techniques [18],[19],[20] LMI [21], integration of the genetic as well as LMI. There are some other techniques to generate Lyapunov: for example, in [22] we proposed numerical techniques that define rational and quadratic Lyapunov functions based on Carleman linearization that permits the computation of the developed DA.

Generally in the literature estimating the domain of attraction is a complex problem. Some famous technique analysis stabilities are Lyapunov and Non-Lyapunov methodologies. The first family, described based on the Lyapunov function, set level associated in the region when a negative sign is included in its time derivative, which can be explained by a mathematical translation pertaining to an elementary observation: when a system's total energy lowers with time, then this system (linear or nonlinear, stationary, or not) tries to revert to an equilibrium state.

Generally, the literature indicates that the Lyapunov theory-based techniques are widely used to estimate the DA [23],[24],[25],[26],[27]. Linear Matrix Inequalities (LMI) optimization was employed by Coutinho et al. To estimate the domain of attraction for dynamic systems by considering the sets of levels of Lyapunov functions [28],[29]. The Largest Approximation of the DoA (LADoA) can be defined based on a Lyapunov Function (LF) for which the local asymptotic stability pertaining to the equilibrium point can be satisfied, characterized by a certain shape, by the LF itself. In such a case, the selection of the LF could significantly impact the conservativeness pertaining to the estimated domain.

C. Literature Review

This study aimed to develop a method for determining the largest approximation DA pertaining to stable equilibrium's by investigating the maximal LFs. The main idea is to identify the best peaks providing an optimal region. The determination of the largest estimation of the DOA via the lyapunov function can be approached by serval method.[55] genetic algorithm will be implemented to adjust the coefficients of lyapunov function and control parameters, in [54] the article aims to develop an original numerical algorithm to construct a polynomial lyapunov function and maximizing domain of attraction using PSO algorithm. With this in mind, a great deal of research has focused on the integration of metaheuristic algorithms as an optimization tool to broaden the domain of attraction. It therefore seems appropriate to provide an overview of the main families of metaheuristic algorithms that can be used in this context. Many problems of optimization can be classified in these categories; Analytic or deterministic; heuristic or random: multi-objective or single objective[16], [30]. As well as we can classify these algorithms, two main classes can be identified: Meta heuristic and gradient. The first class are nature-inspired, and as they have been developed, based on some abstraction of nature. The second class is based on the gradient calculation theory; this class is very complex and risks peeling. In this work, we are interested in the meta heuristic algorithms as they are easy to manipulate and very efficient global search algorithms. Since then, many meta-heuristic nature-inspirited techniques have been developed: the particle swarm optimization (PSO)[49] is an algorithm designed to minic the foraging behavior of brids. Evolutionary strategy (ES)[31], firefly algorithm (FA) [32], ant colony optimization (ACO)[33], differential evolution (DE) [34], probability-based incremental learning (PBIL) [35], big bang–big crunch algorithm [36], bio-geography-based optimization (BBO) [36], harmony search (HS) [37], animal migration optimization (AMO) [38], krill herd method (KH) [39], [40], bat algorithm (BA) [41], teaching learning-based optimization (TLBO) [42], dragonfly algorithm (DA) [43], the Secretary Brid Optimization Algorithm(SBOA) [53]. (SBOA) is an innovative population based meta-heuristic approach such that is inspired from the survival behaviours of secretary brids in their natural habitat. the implementation of SBOA is structured into phases: an exploration step simulating a hunting strategy and an exploitation step simulating a escape strategy. In this work, we are interested by the secretary bird optimization algorithm. The SBOA algorithm offers advantages due to its simplicity showcasing its robustness and wide applicability.

This study employs the secretary birds optimization algorithm (SBOA) to assess the optimal value pertaining to the vertices for optimal domain of attraction (DA) estimation. SBOA is expected to offer higher performance versus the techniques suggested in [2],[12]. This paper aims to ameliorate the technique analyzed in [2] by combining the LMI with SBOA.

The paper is structured as follows: statement problem preliminaries are introduced in Section II, and estimation DA using DAR representation in Section III. In this we further detail the DAR representation and signify the candidate Lyapunov and LMI formulation. In Section IV, we present the Secretary Brids Optimization algorithm implementation. Section V, is reserved for a numerical analysis study. Section VI introduce the discussion of the results, while Section VII ends the paper.

II. Problem Statement and Preliminaries

Given a nonlinear affine systems described by:

$$\left\{ \begin{array}{l} \dot{x} = f(x) + g(x)\, sat(u) \\ u = Kx \end{array} \right. \tag{1}$$

where, $x \in \Re^n$ represents the state variables vector, $u \in \Re$ signifies the control variable, with, $K \in \Re^{1\times n}$can be defined as the assumed constant vector denoting an input gain vector that relies linearly on the system state variables. $f(x), g(x) : \Re^n \to \Re^n$ defines a vectors of a nonlinear value function that satisfies the constraints pertaining to the uniqueness and existence of solution for all $x \in D_x$ and the equilibrium point of interest is the origin. A classical unitary saturation function is defined as follows [2]:

$$sat(u) := sign(u) \min\{|u|, 1\} \tag{2}$$

Analysis of the stability of the system (1) can employ Lyapunov theory [47]. **Lemma** [2]

Assume $V(x)$ to be a Lyapunov function pertaining to the system as Eq. (1) in the following region:

$$J = \{x : V(x) \le 1\} \tag{3}$$

Should $\dot{V}(x)$ be negative, then may be defined as:

$$\dot{V}(x) = \frac{dV(x)}{dt} f(x) \tag{4}$$

The system seemed to be asymptotically stable and for all $x(0) \in J$ the trajectory $x(t)$ corresponds to J while approaching the origin as $t \to \infty$. $\xi(x)$

A. Determining a LF Candidate

This section is dedicated to introduce fundamental outcomes of the Lyapunov theory. Consider the following function:

$$V(x) = \xi(x)^T P \xi(x) \tag{5}$$

where, $\xi(x) \in \Re^{n_\xi}$ represents a rational vector function pertaining to x and $P = P^T \in \Re^{n_\xi \times n_\xi}$ signifies the constant matrix that must be computed. The time derivate pertaining to $V(x)$ has been represented as follows: $\frac{dV(x)}{dt} = \dot{\xi}^T(x) P \xi(x) + \xi^T(x) P \dot{\xi}(x)$. It needs to be noted that $\xi(x)$ and $\dot{\xi}(x)$ denote the rational vector function pertaining to x and $\dot{x}$.

Remark: The Lyapunov function has been presented in [13], which covers a broad class of physical process, including:

- Quadratic LF [8] where; $\xi(x) = x$.
- Bi-quadratic LF [28] and polynomial [29], where $\xi(x)$ being a polynomial vector function in x.
- Rational LF where $\xi(x)$ being a non-singular rational function of x .

As a generally accepted rule, the more complex this vector is, the more conservative the results obtained. Hereafter we assume there exist DAR of $\xi(x)$ and $\dot{\xi}(x)$ defined as follows [2]:

$$\left\{ \begin{array}{l} \xi(x) = E_1 x + E_2 \varsigma(x) \\ 0 = \Gamma_1(x)x + \Gamma_2(x)\varsigma(x) \end{array} \right. \tag{6}$$

$$\left\{ \begin{array}{l} \dot{\xi}(x) = F_1 x + F_2 \chi(x, \dot{x}) \\ 0 = \phi_1(x)\dot{x} + \phi_2(x)\chi(x, \dot{x}) \end{array} \right. \tag{7}$$

where, $\varsigma(x) \in \Re^{n_\varsigma}, \chi(x, \dot{x}) \in \Re^{n_\chi}$ are nonlinear vector functions, $E_1 \in \Re^{n_\xi \times n}, E_2 \in \Re^{n_\xi \times n_\varsigma}, F_1 \in \Re^{n_\xi \times n}, F_2 \in \Re^{n_\xi \times n_\chi}$ are constant matrices, $\Gamma_1(x) \in \Re^{n_\phi \times n}, \Gamma_2(x) \in \Re^{n_\phi \times n_\varsigma}, \phi_1(x) \in \Re^{n_\phi \times n}, \phi_2(x) \in \Re^{n_\phi \times n_\chi}$ are affine matrix functions of x. The representation is called well defined if the following hypotheses satisfied: $\Gamma_2(x), \phi_2(x)$ have full column-rank for all $x \in D_x$.
Using Eq. (6) and Eq. (7) it comes,

$$V(x) = \begin{bmatrix} x \\ \varsigma(x) \end{bmatrix}^T \Delta P \begin{bmatrix} x \\ \varsigma(x) \end{bmatrix} \tag{8}$$

$$\begin{array}{l} \dot{V}(x) = \begin{bmatrix} \dot{x} \\ \dot{\varsigma}(x) \end{bmatrix}^T \Delta P \begin{bmatrix} x \\ \varsigma(x) \end{bmatrix} + \begin{bmatrix} x \\ \varsigma(x) \end{bmatrix}^T \Delta P \begin{bmatrix} \dot{x} \\ \dot{\varsigma}(x) \end{bmatrix} \\ = 2 \begin{bmatrix} x \\ \varsigma(x) \end{bmatrix}^T \Psi P \begin{bmatrix} \dot{x} \\ \chi(x, \dot{x}) \end{bmatrix} \\ \text{with, } \Delta P = \begin{bmatrix} E_1^T P E_1 & E_1^T P E_2 \\ E_2^T P E_1 & E_2^T P E_2 \end{bmatrix}, \\ \Psi P = \begin{bmatrix} E_1^T P F_1 & E_1^T P F_2 \\ E_2^T P F_1 & E_2^T P F_2 \end{bmatrix} \end{array} \tag{9}$$

B. Domain of Attraction

Domain of attraction (DA) can be described as those initial conditions in which the states converge towards equilibrium asymptotically [44], [46], [45]. According to the Lyapunov function introduced in the Section II-A we can estimate the domain of attraction. A region of attraction is given as follows;

$$J = \left\{ x \in D_x : \xi(x)^T P \xi(x) \le 1 \right\} \tag{10}$$

where, $P \in \Re^{n\times n}$ is a positive definite matrix and J is the normalised ellipsoid. When the condition $\dot{V}(x)$ hold for all $x(0) \in J$, the region J can be represent an estimated domain of attraction for system (1), this means that any trajectory starting within J will converge to the origin, without exiting J.

C. Statement of the Generalized Sector-Based Constraint

Consider $H \in \Re^{1\times n}$ the row vector function and define the set below:

$$S = \{x \in \Re^n; |(K-H)x| \leq 1\} \tag{11}$$

when x belongs to S, the relation can be stated as [5], then the deadzone nonlinearity $\psi(Kx)$ meets the following inequality which is valid for any positive scalar μ.

$$\psi(Kx)^T \mu\left[\psi(Kx) - Hx\right] \leq 0 \tag{12}$$

Let the set J and S defined respectively in (10) and (11) and consider a matrix $C(x) \in \Re^{n_c\times(n+n_\xi)}$ affine in x, where $C(x)\left[\begin{array}{cc} x & \varsigma(x)\end{array}\right] = 0$. Let S included in J. If a matrix Ξ exists the following condition is satisfied:

$$\left[\begin{array}{cc} 1 & \left[\begin{array}{cc}(K-H) & 0\end{array}\right] \\ \left[\begin{array}{c} K^T - H^T \\ 0\end{array}\right] & \left(\sum(P) + NC(x) + C(x)^T N^T\right)\end{array}\right] \geq 0 \tag{13}$$

D. Polytope in State Space

Let D_x is given polytope (with n_e vertices), which defines the intiales conditions and contains the origin. Therefore, the polytope can be defined as follows:

$$D_x = \left\{x \in \Re^n : a_i^T x \leq 1, i = 1, ..., n_e\right\} \tag{14}$$

with, the constant vectors $a_i \in \Re^n$ are defined such that $a_i^T x = 1$ for all groups of adjacent vertices.
Similarly, to the result of section II-C the set J include in D_x and a matrix $C(x)\left[\begin{array}{cc} x & \varsigma(x)\end{array}\right] = 0$, if the following condition is satisfied:

$$\left[\begin{array}{cc} 1 & \left[\begin{array}{cc}-a_i^T & 0\end{array}\right] \\ \left[\begin{array}{c} -a_i \\ 0\end{array}\right] & \left(\Sigma(P) + RC(x) + C(x)^T R^T\right)\end{array}\right] \geq 0, \quad \forall k \in \{1, \ldots, n_e\} \tag{15}$$

III. Estimation of the Domain of Attraction using DAR Representation

A. Differential Algebraic Representation DAR

Set the nonlinearity of the following dead zone [48]

$$\psi(u) = u - sat(u) \tag{16}$$

A nonlinear dead-zone $\psi(u)$ is defined in this work to justify the occurrence of the saturation nonlinearity. Taken into accounts, Eq. (16) the system Eq. (1) is presented as follows:

$$\left\{\begin{array}{l} \dot{x} = f(x) + g(x)u - g(x)\psi(u) \\ u = Kx \end{array}\right. \tag{17}$$

A nonlinear system can be described in many different representation. In this case, the system is represented by the differential algebraic representation (DAR). That is defined as follows:

$$\left\{\begin{array}{l} \dot{x} = A_1 x + A_2 z(x) + A_3 sat(Kx) \\ 0 = \pi_1 x + \pi_2 z(x) + \pi_3 sat(Kx) \end{array}\right. \tag{18}$$

where $z \in \Re^{n_z}$ signifies a nonlinear auxiliary vector pertaining to x, which includes the nonlinear elements in $f(x)$. $A_1 \in \Re^{n\times n}$, $A_2 \in \Re^{n\times n_z}$ and $A_3 \in \Re^{n\times 1}$ can be defined as constant matrices, and $\pi_1 \in \Re^{n_z\times n}$,$\pi_2 \in \Re^{n_z\times n_z}$,$\pi_3 \in \Re^{n_z\times n}$ represent the affine matrix functions pertaining to x. To ensure the differential algebraic representation is well determined and the solution x is unique, the previous assumptions have been implemented. If z is invisible, considering Eq. (18) we have that:

$$z(x) = -\pi_2^{-1}\left(\pi_1 x + \pi_3 sat(u)\right) \tag{19}$$

System (1) can be expressed as follows:

$$\dot{x} = (A_1 - A_2\pi_2^{-1}\pi_1)x + (A_3 - A_2\pi_2^{-1}\pi_3)sat(u) \tag{20}$$

As a result, the term $sat(u)$ can be substituted with Eq. (16), so the system Eq. (1) can be expressed in the following form:

$$\left\{\begin{array}{l} \dot{x} = (A_1 + BK)x + A_2 z(x) - B\psi(Kx) \\ 0 = (\pi_1 + \pi_3 K)x + \pi_2 z(x) - \pi_3\psi(Kx) \end{array}\right. \tag{21}$$

with, $B = A_3 \in \Re^{n\times 1}$.

B. LMI Formulation

In this part of the study, we made a proposition for LMI condition development to ensure the Lypaunov function presented in Eq. (8). This LMI is attained by integrating the linear annihilator condition [10] and Finsler's lemma [47]. Thus, it is possible to define the solution of estimating the area of attraction with respect to LMIs that are state dependent as shown in the theorem given below.

For the system represented in Eq. (21), the Lypaunov function in Eq. (8) and its time derivative in Eq. (9) are considered first with, $\nu = \left[\begin{array}{cccc}\xi & \varphi & z & \psi(Kx)\end{array}\right]$

Than it comes:

$$\frac{dV(x)}{dt} = \left[\begin{array}{c}\xi \\ \varphi \\ z \\ \psi(Kx)\end{array}\right]\left[\begin{array}{cccc} 0 & \Psi P & 0 & 0 \\ \Psi P^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0\end{array}\right]\left[\begin{array}{c}\xi \\ \varphi \\ z \\ \psi(Kx)\end{array}\right]^T < 0 \tag{22}$$

According to the Section II-C if the relation in Eq. (12) is verified for any positive scalar such that

$$\frac{dV(x)}{dt} - 2\psi(Kx)^T \mu(\psi(Kx) - Qx) < 0 \tag{23}$$

Therefore (23) can be written as follows

$$\nu^T \Lambda \nu < 0 \tag{24}$$

$$\text{with, } \Lambda = \begin{bmatrix} 0 & \Psi P & 0 & \begin{bmatrix} \mu H^T \\ 0 \end{bmatrix} \\ \Psi P^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \begin{bmatrix} \mu H & 0 \end{bmatrix} & 0 & 0 & -\mu \end{bmatrix}$$

by exploiting the DAR's property of equality in Eq. (21), Eq. (6), Eq. (7) and the linear annihilator obtained by the formula [10] can be utilized to obtain that

$$\begin{bmatrix} L(x) & 0 & 0 & 0 & 0 & 0 \\ \Gamma_1(x) & \Gamma_2(x) & 0 & 0 & 0 & 0 \\ 0 & 0 & \phi_1(x) & \phi_2(x) & 0 & 0 \\ A_{cl} & 0 & -\mathrm{I}_n & 0 & A_2 & -B \\ \pi_1(x) + K\pi_3(x) & 0 & 0 & 0 & \pi_2(x) & \pi_3(x) \end{bmatrix} \begin{bmatrix} x \\ \varsigma(x) \\ \dot{x} \\ \chi(x,\dot{x}) \\ z \\ \psi(x) \end{bmatrix} = 0 \tag{25}$$

with, $A_{cl} = A_1 + BK$.

Using the Finsler's lemma[47] for Eq. (24), Eq. (25) such that:

$$\begin{bmatrix} 0 & \Psi P & 0 & \begin{bmatrix} \mu H^T \\ 0 \end{bmatrix} \\ \Psi P^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \begin{bmatrix} \mu H & 0 \end{bmatrix} & 0 & 0 & -\mu \end{bmatrix} + MX(x) + X^T(x) M^T < 0, \tag{26}$$

$$\text{with, } L(x) = \begin{bmatrix} x_2 & -x_1 & 0 & \cdots & 0 \\ 0 & x_3 & -x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & x_n & -x_{(n-1)} \end{bmatrix},$$

$$C(x) = \begin{bmatrix} L(x) & 0 \\ \Gamma_1(x) & \Gamma_2(x) \end{bmatrix}$$

Theorem 1

Consider the nonlinear system with saturating actuators given in Eq. (1) with $u(t) = Kx(t)$ and let $\xi(x)$ be a rational vector function in terms of x with DAR of $\xi(x)$ and $\dot{\xi}(x)$ as given in Eq. (6) and Eq. (7) and lets define a polytope D_x. If there exists matrices $P = P^T, H, R, N, M$ and Υ, of appropriate dimensions that satisfy the following LMIs for all $x \in \Theta(D_x)$. For instance, the following LMIs can be represented:

$$\Delta P + \Upsilon C(x) + C(x)^T \Upsilon^T > 0 \tag{27}$$

$$\begin{bmatrix} 1 & \begin{bmatrix} -a_i^T & 0 \end{bmatrix} \\ \begin{bmatrix} -a_i \\ 0 \end{bmatrix} & \Delta P + RC(x) + C^T(x) R^T \end{bmatrix} \geq 0 \tag{28}$$

$$\begin{bmatrix} 1 & \begin{bmatrix} K - H & 0 \end{bmatrix} \\ \begin{bmatrix} K^T - H^T \\ 0 \end{bmatrix} & \Delta P + NC(x) + C^T(x) N^T \end{bmatrix} \geq 0 \tag{29}$$

$$\begin{bmatrix} 0 & \psi P & 0 & \begin{bmatrix} H^T \mu \\ 0 \end{bmatrix} \\ \psi P & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \begin{bmatrix} H\mu & 0 \end{bmatrix} & 0 & 0 & -\mu \end{bmatrix} + M\mathrm{X}(x) + \mathrm{X}^T(x) M^T < 0 \tag{30}$$

So, for all $x(0) \in J$, trajectory $x(t)$ belongs to J and $V(x) = \xi(x)^T P \xi(x)$ in Eq. (5) is a Lyapunov function in D_x. Theorem 1 establishes a sufficient condition to guarantee that an J formed by a Lyapunov function, is an domain of asymptotic stability for the closed loop system. Also, let try to find a domain estimate as large as possible. Therefore, the idea is to select among all possible feasible solutions for LMI in Eq. (27)-(30) the one that offers the largest possible set J, taking into account a volume size criterion. In the more general case described here, where the domain of attraction is an ellipsoid, a solution to the volume maximization problem can be directly addressed through the following optimization problem:

$$\begin{cases} Max\,(Vol) = \frac{1}{trace(P)} \\ subject\ to : (27) - (30) \end{cases} \tag{31}$$

The domain of attraction volume maximization cannot exist when parameters introduced in LMI are not well chosen. For example if we leave μ free, or badly chosen we lose the convexity of the conditions of Theorem 1 and consequently LMI does not give solutions from where it will not be feasible. On the other hand, the polytope search of admissible states gives the largest domain of the state space $D_x(a_1, a_2)$ so that the system is stable by solving Eq. (27-30), for an optimal pair (a_1, a_2). In the end the attraction domain search will be bounded in the admission polytope by selecting the optimal Lyapunov function.

IV. Main Results

In this section we develop an algorithm to expand the domain of attraction using the SBOA-based meta-heuristic method. This section introduces a strategy for selecting the right parameter μ to guarantee the convexity of theorem 1, the best pair (a_1, a_2) to maximize the admissible polytope of the states and the right choice of the optimal Lyapunov function matrix P. Evalutionary tehniques are emloyed to determine the parameters μ, and the coefficients of the Lyapunov function. The form of the candidate Lyapunov function defined by the user, and its corresponding domain are validated for and the LMIs of theorem 1, are feasible. In case these conditions are not met and the LMIs are not feasible, we repeatedly estimate the parameters in question and the coefficients of the Lyapunov function until the LMI optimization has a solution. The basis of this criterion is to obtain optimal parameters through the use of evolutionary algorithms.

A. Proposed Method to Enlarging DA

1) Secretary Birds Optimization Algorithm: Secretary birds [53] are large, terrestrial birds of prey that generally found in tropical savannas or semi-desert regions. Secretary birds are perdators of snakes on the Arican continent including species like the black mamba and Cobra. The SBOA algorithm derives inpiration from the survival techniques of searching or prey and escaping tough ecologies. A secretary birds optimisation algorithm is created to handle complex optimisation challenges. The secretary brid's itelligence is showcased in its strategies for evading predators. The SBOA is composed of the following parts:

Initiation Phase: The first step in solving a typical minimization problem $f(x)$ is to determine the initial solutions that will be used to initiate the search. In the population, each individual represents a solution to the optimization problem and this solution is initialized according to the following equation:

$$X_j = lb + r \times (ub - lb)\,, j = 1, 2, \ldots, N \tag{32}$$

where, X_j is the postion of the j^{th} lb and ub are the lower and upper bounds. $rand$ designates a random number in $[0, 1]$. N is the dimension of the problem. In addition, the fitness value of the solution X_i denoted as $F_i = f(X_i)$ is a measure of its quality.

Hunting Strategy of Secretary Birds (Exploration step): Contraty to the other fierce predators secretay birds employ a more intelligent strategy for hunting snakes. Therefore, the whole hunting process can be broken down into three steps, they include searching prey, consuming prey, and attacking prey.

- Searching prey: In this stage, secretary birds need seek prey while keeping a safe range. By referencing the positions of the others two secretary birds, the secretary brid can scout new potential areas. For this reason, the differential mutation operations are incoperated to maintain algorithm diversity. The position of each individual X_i is updated using equations (33) and (34) when the current iteration time t is smaller than one-third of the maximum iterations T.

$$X_t^{newP}(t) = X_t(t) + (X_{random_1}(t) - X_{random_2}(t)) \times R_1 \tag{33}$$

$$\begin{cases} X_t(t+1) = X_t^{newP}(t)\,, \quad if \quad F_t^{newP} < F_i \\ X_t(t+1) = X_t(t)\,, else \end{cases} \tag{34}$$

Where, R_1 is a random vector consisting of $1 \times M$ elements chosen randoml from $[0, 1]$. $X_{random_1}, X_{random_2}$ represent two individuals randomly chosen from the present population.

- Consuming prey: When secretary brids identify potential prey their first action is to hover around the snake exhibiting aile footwork and maneuvers. Through observing and baiting opponents while circling, the prey's patience will be exhausted, causing it to lower its guard. Considering the current best individual as the prey, other secretary birds adjust their positions to move closer to it. In this stage, we use Brownian motion (RB) to simulate a random movement of the secretary birds such that is given by:

$$RB = randn(1, D) \tag{35}$$

Hence, updating the secretary birds position during the consuming pery stage can be represented as follows:

$$X_t^{newP}(t) = X_{best}(t) + e^{\left(\frac{t}{T}\right)^4} \times (RB - 0.5) \times (X_{best}(t) - X_t(t)) \tag{36}$$

$$\begin{cases} X_t(t+1) = X_t^{newP}(t)\,, \quad if \quad F_t^{newP} < F_t \\ X_t(t+1) = X_t(t)\,, else \end{cases} \tag{37}$$

where, X_{best} represent the best solution for the current population.

- Attacking prey: When, the prey well exhausted, so it the time to start the attack. Here the secretary brid used the Levy flight approach. Therefore, the characteristics of this stage are describe by the following Eq. (38), (39)

$$X_t^{newP}(t) = X_{best}(t) + \left(\left(1 - \frac{t}{T}\right)^{\frac{2\times t}{T}}\right) \times X_t(t) \times RL \tag{38}$$

$$\begin{matrix} X_t(t+1) = X_t^{newP}(t)\,, if \quad F_t^{newP} < F_t \\ X_t(t+1) = X_t(t)\,, else \end{matrix} \tag{39}$$

where RL represents a random movement(the levy fight representation) which is defined as follows:

$$RL = 0.5 \times Levy(M) \tag{40}$$

Escape Strategy for Secretary Birds (Exploitation step): In nature the main enemies of secretary bird are large predators. Such as eagles, hawks, foxes, and jackals, which may attack them or steal their food. In this cas we proposed two categories:

- Camouflage based on environment: When secretary birds are confronted by enemies, their first strategy is to camouflage themselves in order to avoid danger. The secretary birds modify their positions around the prey (which represents the best individual) reflecting the behavior of attempting to evade local optimal algorithms. The following Eq. (41),(42) present the mathematical model of this approach.

$$\mathrm{X}_t^{newP}(t) = \mathrm{X}_{best}(t) + (2 \times RB - 1) \times \left(1 - \frac{t}{T}\right)^2 \times \mathrm{X}_t(t) \tag{41}$$

$$\begin{cases} X_t(t+1) = X_t^{newP}(t)\,, if \quad F_t^{newP} < F_t \\ X_t(t+1) = X_t(t)\,, else \end{cases} \tag{42}$$

Where, $\left(1 - \frac{t}{T}\right)^2$ is a disturbance factor that helps to strike a balance between exploration (seeking new solutions) and exploitation (using known solutions).

- Running mode: Then in this step, if they can't avoid they enem we use the approach of flight or rapid running to maintain their saftey. Secretary brid updated their new position by using the following equation

$$X_t^{newP}(t) = X_{best}(t) + R_2 \times (X_{rand}(t) - K \times X_t(t)) \tag{43}$$

$$\begin{cases} X_t(t+1) = X_t^{newP}(t), if \quad F_t^{newP} < F_t \\ X_t(t+1) = X_t(t), else \end{cases} \tag{44}$$

In a SBOA, the main operation involves computing the fitness function. Therefore the quality of the particle is evaluated using its objective function, the goal being to maximize it.

$$v = \frac{1}{trace(P)} \tag{45}$$

To achieve this objective, a meta-heuristic technique is used. We developed a technique to expand the DA by integrating the SBO algorithm and an LMI technique to ensure the computing of the maximal LF defined in Eq. (4). In this work we estimate again by SBOA μ and the values of the vertices a_1 and a_2 to expand the domain of attraction defined by Eq. (10).

$$\begin{cases} Max(Vol(J)) = \frac{1}{trace(P)} \\ s.t: \begin{cases} \min \theta \\ s.t: LMI \ (27)-(30), \ a \quad feasible \quad solution \\ \theta - trace\left(\Delta P + \Upsilon C(x) + C(x)^T \Upsilon^T\right) > 0 \end{cases} \end{cases} \tag{46}$$

The designed approach is synthesized in the following flowchart depicted in the Fig. 1.

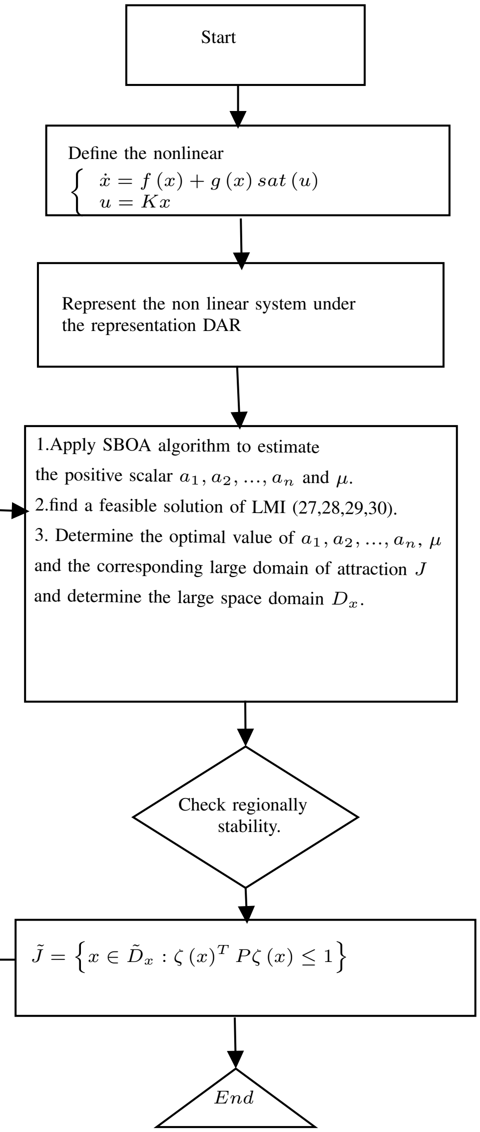


Fig. 1. Flowchart of the meta-heuristic technique for estimating the DA of a nonlinear system with input saturation.

V. Numerical Examples

In this section, numerical examples are presented to verify the effectiveness of the proposed approach. The conditions introduced in this paper were implemented in MATLAB (R2015) using the parser Yalmip and the solver SDPT3.

Example 1

Consider a nonlinear system with saturated input given by [2]:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = (1 + x_1^2)x_1 + (2 + 8x_2^2)x_2 + sat(u) \\ u = Kx \end{cases} \tag{47}$$

The state is $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ and the control input is $u(t) = -2x_1(t) - 4x_2(t)$. In this paper, we search for the Lyapunov function that stabilizes the system asymptotically, minimizes the cost function represented by the trace of Matrix P definite positive, and has the largest estimation of the DA of system (47) in a closed loop. We use the proposed method for example 1. Note that, $D_x(a_1, a_2) := \left\{x \in \Re^2 : |x_1| \leq a_1, |x_2| \leq a_2\right\}$ is a state admissible, with a_1, a_2 tow positive predefined scalars. First, we reformulate the system (47) in the DAR form. Then, by applying the equation (16)-(18), a DAR of this system is given by:

$$\begin{cases} \dot{x} = (A_1 + BK)x + A_2 z(x) - B\psi(Kx) \\ 0 = (\pi_1 + \pi_3 K)x + \pi_2 z(x) - \pi_3 \psi(Kx) \end{cases} \tag{48}$$

where, $z = \begin{bmatrix} x_1^2 & x_2^2 & x_1^3 & x_2^3 \end{bmatrix}^T$, $A_{cl} = A_1 + BK$, $A_3 = B$, $A_{cl} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 8 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

$$\pi_1 = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \pi_2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ x_1 & 0 & -1 & 0 \\ 0 & x_2 & 0 & -1 \end{bmatrix}, \pi_3 = 0$$

As a means of evaluating the stability of the system, two distinctive Lyapunov functions are taken. First, the quadratic Lyapunov function is analyzed:

$$V_1(x) = \xi_1^T(x) P_1 \xi_1(x) \tag{49}$$

Second a polynomial Lyapunov function is considered:

$$V_2(x) = \xi_2^T(x) P_2 \xi_2(x) \tag{50}$$

where $\xi_1(x) = x$, $\xi_2(x) = \begin{bmatrix} x_1^2 & x_1x_2 & x_2^2 & x_1 & x_2 \end{bmatrix}^T$. $P_1 \in \Re^{2\times 2}$ and $P_2 \in \Re^{5\times 5}$ are two symmetric matrices to be computed. The polynomial LF calculation requests the decomposition of $\xi_2(x)$ and $\dot{\xi}_2(x)$ as stated below.

$$\begin{cases} \xi_2(x) = E_1 x + E_2 \varsigma(x) \\ 0 = \Gamma_1(x)x + \Gamma_2\varsigma(x) \end{cases}, \quad \begin{cases} \dot{\xi}_2(x) = F_1\dot{x} + F_2\chi(x,\dot{x}) \\ 0 = \phi_1(x)\dot{x} + \phi_2\chi(x,\dot{x}) \end{cases} \tag{51}$$

where, $\varsigma(x) = \begin{bmatrix} x_1^2 & x_1x_2 & x_2^2 \end{bmatrix}^T$, $E_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$,

$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\Gamma_1(x) = \begin{bmatrix} x_1 & 0 \\ 0 & x_1 \\ 0 & x_2 \end{bmatrix}$,

$\Gamma_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $\chi(x,\dot{x}) = \begin{bmatrix} x_1\dot{x}_2 \\ x_1\dot{x}_2 \\ x_2\dot{x}_1 \\ x_2\dot{x}_2 \end{bmatrix}$,

$F_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, $F_2 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$,

$\phi_1(x) = \begin{bmatrix} x_1 & 0 \\ 0 & x_1 \\ x_2 & 0 \\ 0 & x_2 \end{bmatrix}$, $\phi_2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

the optimization problem (46) was solved to obtain the largest admissible polytope in state space and that maximizes the domain of attraction. The Fig. 2 shows the dynamic of the cost function. The optimal domain is obtained as:

$$\begin{cases} (a_1, a_2) = \begin{bmatrix} 0.51233 & 0.32322 \end{bmatrix} \\ P_{opt} = \begin{bmatrix} 4.6221 & 3.0714 \\ 3.0714 & 11.6130 \end{bmatrix}, \\ \theta = 16.2351 \end{cases} \tag{52}$$

Fig. 3 shows the evolution of the DA for the system Eq. (47) using SBOA approach. Fig. 4 represents the dynamic of the state space initialized from the tangency point state locus and the evolution of the input control of the system.

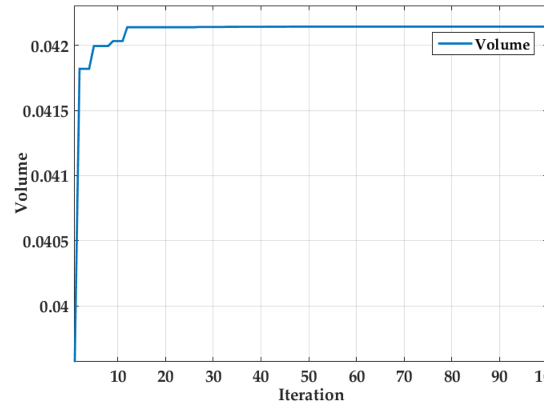


Fig. 2. Dynamic of the cost function using the SBOA approach for system (47) for obtain the optimal value of μ.

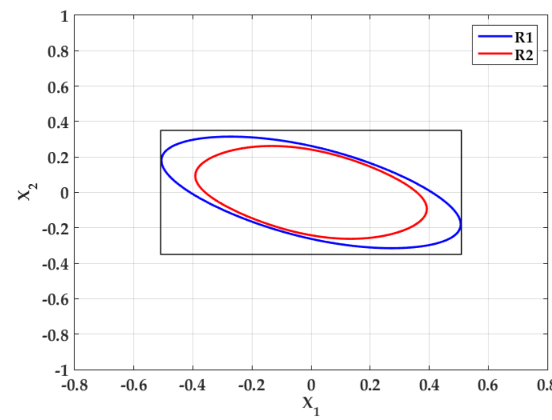


Fig. 3. Approximation of the DA for the system Eq. (47) using quadratic LF (Blue ellipsoid using SBOA technique - red ellipsoid analytical technique).

Example 2

Considering a single-link robot arm in [50]

$$\ddot{\theta}(t) = -\frac{Mgl}{J}\sin(\theta) - \frac{D}{J}\dot{\theta} + \frac{1}{J}u \tag{53}$$

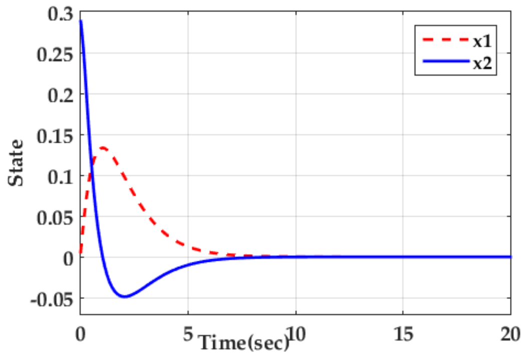
where,

- θ is the angle position of the arm
- u is the input control.
- M is the mass of the poyload.
- J is the moment of inertia.
- g is the acceleration of gravity and l is the length of the arm.

Assume, $x_1 = \theta, x_2 = \dot{\theta}$ then we obtain the following state space model of the robotic arm manipulator

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{Mgl}{J}\sin(x_1) - \frac{D}{J}x_2 + \frac{1}{J}u \\ y = x \end{cases} \tag{54}$$

Then the values parameters for robotic arm are given by: $g = 9.81$, $l = 0.5$, for this work we consider the nominal value of $D =$



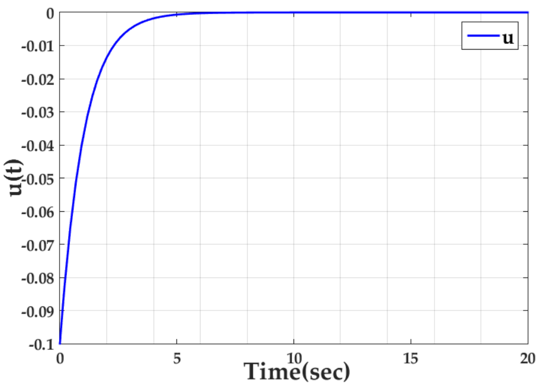


Fig. 4. Results of simulations for example 1: states dynamics and control input.

$D_0 = 2$ and for J and M we consider the mode 1 so $M = J = 1$, With this parameters chosen we obtain the following representation.

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -4.905 \times \sin(x_1) - 2 \times x_2 + u \\ y = x \end{cases} \tag{55}$$

Using the Taylor series at$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ then in this work we stop at $n = 3$ therefore $\sin(x) = x - \frac{x^3}{3!}$. Then, we obtain the following state-space:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -4.905 \times x_1 + \frac{4.905}{6} \times x_1^3 - 2 \times x_2 + u \\ y = x \end{cases} \tag{56}$$

This example cannot be applied with this approach because the absence of any indication for the command u. Where, $u = k_1 x_1 + k_2 x_2$ to find the parameters of the command u we want to apply Chesi 2004 [51]. Therefore, we linearize in the vicinity of the equilibrium to establish a Lyapunov function LF:

$$A = \frac{\partial f}{\partial x} \Big\|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -4.905 & -2 \end{bmatrix} \tag{57}$$

Then, for determine the matrix P we used this equation: $A^T P + PA = -Q$ with $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Such that, we obtain the following matrix P

$$P = \begin{bmatrix} 1.68 & 0.102 \\ 0.102 & 0.301 \end{bmatrix} \tag{58}$$

with $V(x) = 1.68x_1^2 + 0.204x_1x_2 + 0.301x_2^2$ and the controller class defined by $\phi(y) = y$, $\Im = \left\{U = \begin{bmatrix} u_1 & u_2 \end{bmatrix} : u \in \begin{bmatrix} -1 & 1 \end{bmatrix}\right\}$. Let us examine the structure of the GEVP introduced in [51]. The degree of $\dot{V}$ denoted as δ_d is 4. As a result, δ_s can be chosen as follows: $\delta_s = 1$, so $m = 2$ and $x^{\{\delta_v\}} = x^{\{\delta_s\}} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T, x^{\{m\}} = \begin{bmatrix} x_1x_2x_1^2x_1x_2x_2^2 \end{bmatrix}^T$, it is found that

$$D_f(\alpha) = \begin{bmatrix} -1.0006 & -4\times10^{-4} & 0 & \alpha_1 & \alpha_2 \\ -4\times10^{-4} & -1 & -\alpha_1 & -\alpha_2 & 0 \\ 0 & -\alpha_1 & 0.2501 & 0.2460 & \alpha_3 \\ \alpha_1 & -\alpha_2 & 0.2460 & -2\alpha_3 & 0 \\ \alpha_2 & 0 & \alpha_3 & 0 & 0 \end{bmatrix},$$

$$D_g(U) = \begin{bmatrix} 0.204u_1 & 0.301u_1 + 0.102u_2 & 0 & 0 & 0 \\ 0.301u_1 + 0.102u_2 & 0.602u_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In order to compute $W(S)$, let's note that $V = I_2$. Therefore, we have that $s = \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}$,

$$K = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$W(S) = \begin{bmatrix} s_1 & s_2 & 0 & 0 & 0 \\ s_2 & s_3 & 0 & 0 & 0 \\ 0 & 0 & \tau s_1 & \tau s_2 & 0 \\ 0 & 0 & \tau s_2 & \tau(s_1 + s_3) & \tau s_2 \\ 0 & 0 & 0 & \tau s_2 & \tau s_3 \end{bmatrix}$$

$$W_2(S) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.68s_1 & 0.84s_2 + 0.102s_1 & 0 \\ 0 & 0 & 0.84s_2 + 0.102s_1 & 1.68s_3 + 0.408s_2 + 0.301s_1 & 0.102s_3 + 0.302s_2 \\ 0 & 0 & 0 & 0.102s_3 + 0.302s_2 & 0.301s_3 \end{bmatrix}$$

The obtained input control u is given by: $u = -0.5043x_1 - 0.2863x_2$. In this step, we want to apply our proposal and given the DAR representation Eq. (21) with the auxiliary vector $z = x_1^2$, we consider

$A_1 = \begin{bmatrix} 0 & 1 \\ -5.4093 & -2.2863 \end{bmatrix}, A_2 = \begin{bmatrix} 0 \\ x_1 \end{bmatrix}, A_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \pi_1 = \begin{bmatrix} x_1 & 0 \end{bmatrix}, \pi_2 = -1$ and $\pi_3 = 0$.

To analyze the stability, we consider a quadratic Lyapunov function. SBOA is implemented with (maximum number of iterations 100 and the swarm size is 30). To obtain the optimal value of μ, (a_1, a_2). Fig. 5 present the evolution of the SBOA process for 100 iterations.

$$\begin{cases} (a_1, a_2) = \begin{bmatrix} 1.4221 & 2 \end{bmatrix} \\ P_{opt} = \begin{bmatrix} 0.6106 & 0.1142 \\ 0.1142 & 0.2714 \end{bmatrix} \\ \theta = \ 0\ .8819 \end{cases} \quad (59)$$

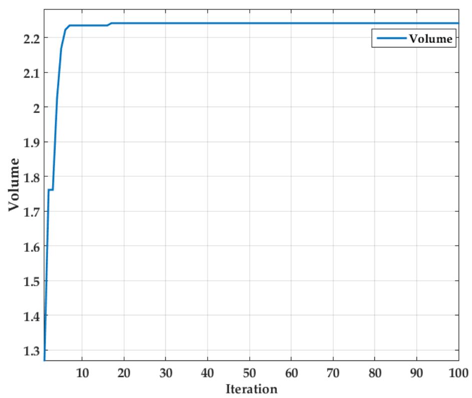


Fig. 5. Dynamic of cost function volume using the SBOA approach for nonlinear system to obtain the optimal value of μ.

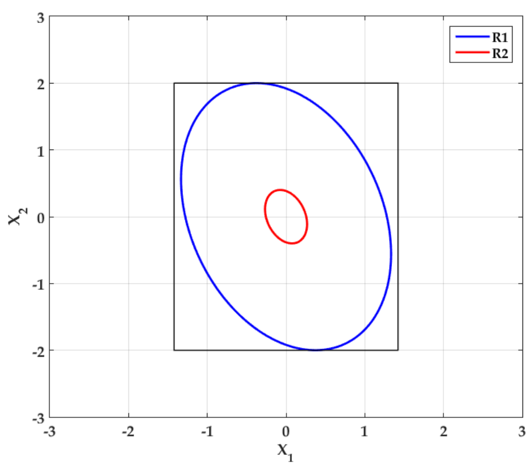


Fig. 6. Estimation of the DA for system Eq. (54) using quadratic LF (Blue ellipsoid using SBOA technique-Red ellipsoid using analytical technique).

The largest DA is represented by the blue ellipsoid in Fig. 6. Fig. 7 presents the evolution of the state space and the input control.

VI. Discussion

This section seeks to compare the advantages of the current work's strategy with those of works presented in study [12] and study [2] as part of a bench-marking study. Table I summarizes the different results achieved for four dynamic nonlinear systems taking into account input saturation. The implementation of the SBOA method incorporates both quadratic Lyapunov functions. In this study, obtaining the maximum volume of the region of attraction serves as the primary criterion for evaluation. As far as the domain of attraction values are concerned, the results obtained are clearly superior. DA features are shown in table I for four nonlinear dynamic systems with quadratic Lyapunov functions. Systems of E1, E2 and E3 are second-order systems. However, E4 is a nonlinear third order system.

TABLE I. Summary of the Bench-Marking Comparative Study

Example	System dynamic	Lyapunov Function	Estimation of DA		Estimation of DA with SBOA	
			Polytope	Volume	Volume	Polytope
E1[12]	$\begin{cases} \dot{x}_1 = -2x_1 + x_1x_2 \\ \dot{x}_2 = x_1 + x_2 + x_1x_2 + sat(u) \end{cases}$	Quadratic	$\lvert x_1\rvert \leq 0.7,$ $\lvert x_2\rvert \leq 0.7$	$v = 0.2421$	$\lvert x_1\rvert \leq 1.1889,$ $\lvert x_2\rvert \leq 0.80046$	$v = 0.3251$
E2[12]	$\begin{cases} \dot{x}_1 = \frac{1+x_1^2}{2}x_2 \\ \dot{x}_2 = \frac{2}{1+x_1^2}x_1 - x_2 - \frac{1-x_1^2}{1+x_1^2}sat(u) \end{cases}$	Quadratic	$\lvert x_1\rvert \leq 0.4,$ $\lvert x_2\rvert \leq 0.6$	$v = 0.1024$	$\lvert x_1\rvert \leq 0.44745,$ $\lvert x_2\rvert \leq 0.94678$	$v = 0.1116$
E3 [52]	$\begin{cases} \dot{x} = \begin{bmatrix} 01 \\ -\frac{c_3}{M} & -\frac{c_1+c_2x_1^2}{M} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1+c_4x_2^2}{M} \end{bmatrix}u, \\ y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{cases}$	Quadratic	$\lvert x_1\rvert \leq 0.45,$ $\lvert x_2\rvert \leq 0.32$	$v = 0.0679$	$\lvert x_1\rvert \leq 01.5,$ $\lvert x_2\rvert \leq 1.4018$	$v = 1.0489$
E4[28]	$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \varepsilon_1x_2 + \varepsilon_2\tau_1 + \eta_1 \\ \dot{x}_3 = \cos x_1 x_2 \\ 0 = \tau_1 - \varepsilon_4x_3 - \varepsilon_5\cos x_1 \\ 0 = x_3^2 + \cos x_1^2 - 1 \end{cases}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$	Quadratic	$\lvert x_1\rvert \leq 1.1$ $\lvert x_2\rvert \leq 1.1$ $\lvert x_3\rvert \leq 8$	$v = 0.0769$	$\lvert x_1\rvert \leq 1.4984$ $\lvert x_2\rvert \leq 1.5$ $\lvert x_3\rvert \leq 2$	$v = 0.2033$

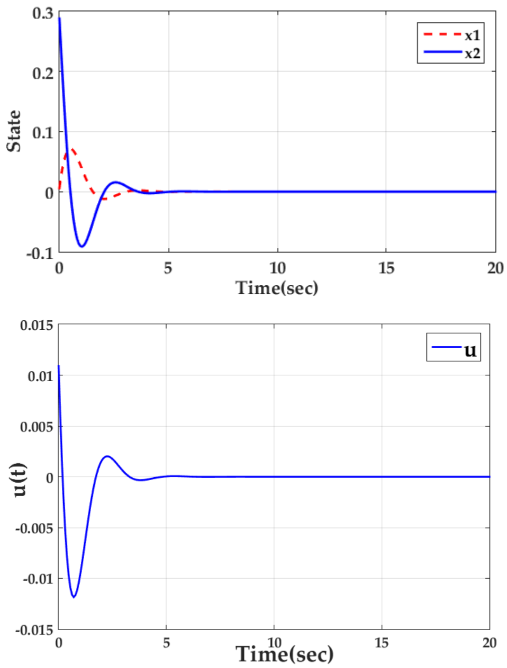


Fig. 7. Simulation results of example 2: states and input control.

The example 5.1 illustrates the advantage of a polynomial Lyapunov function over a quadratic Lyapunov function. Please be aware that the domain of volume serves as the primary evaluation factor. For each example, going through three stages using the SBOA approach, and comparing the results obtained by the technique presented in study [2].

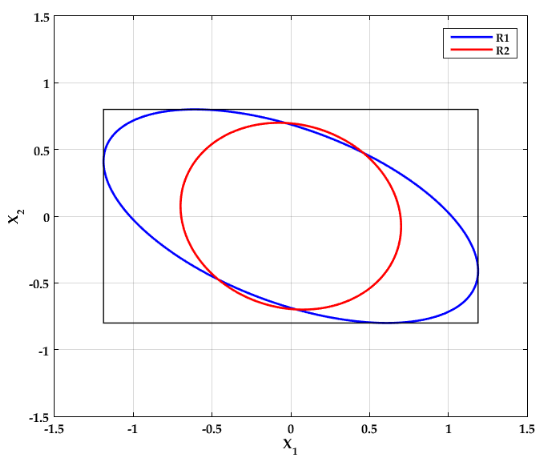


Fig. 8. DA that is approximation based on the SBOA approach for the origin in case E1 that is listed in Table I.

Fig. 8, 9, 10 and 11 illustrates the approximate domain of attraction for example E1-E4 described in Table I. The blue ellipsoid illustrates the estimated DA as determined by the SBOA method, while the red ellipsoid illustrates the estimated DA as determined by the approach described in study [2]. As a result, Fig. 12, 13, 14 and 15 depict the dynamic of state variables and their control inputs for each of the examples in Table I. A stable equilibrium point can be established asymptotically by the state variables. In this regard, the strategy developed offers a complete solution that begins by guaranteeing the local stability of the system, then estimates the states admissible, and finally provides the maximal domain volume possible.

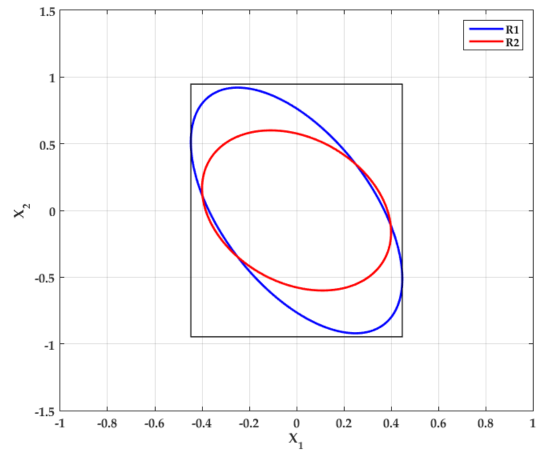


Fig. 9. DA that is approximation based on the SBOA approach for the origin in case E2 that is listed in Table I.

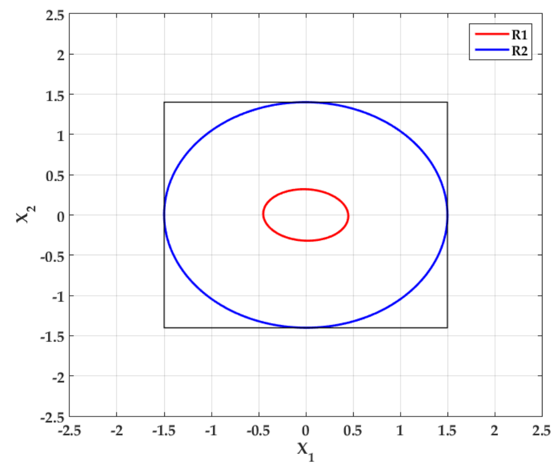


Fig. 10. DA that is approximation based on the SBOA approach for the origin in case E3 that is listed in Table I.

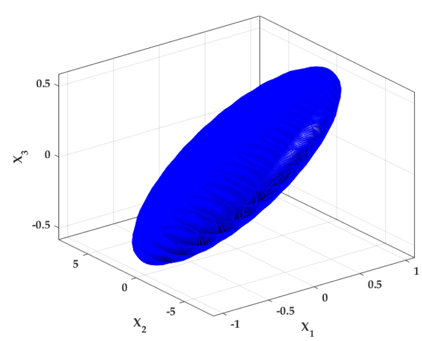


Fig. 11. DA that is approximation based on the SBOA approach for the origin in case E4 that is listed in Table I.

Furthermore, Table I clearly illustrates that, for the four examples studied, the SBOA technique yields significantly better estimates of the basin of attraction than the approach presented in study [2].

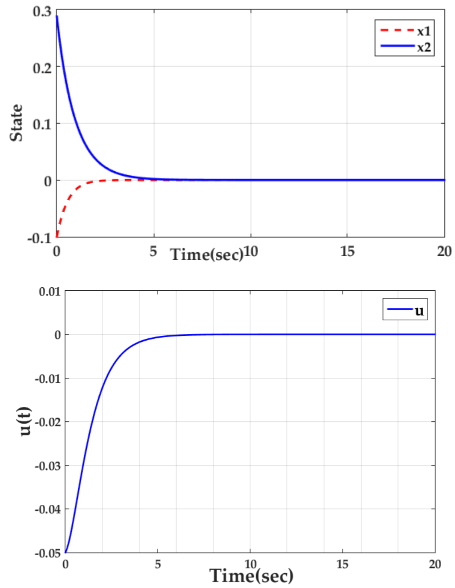


Fig. 12. Simulation results of example E1: states and input control.

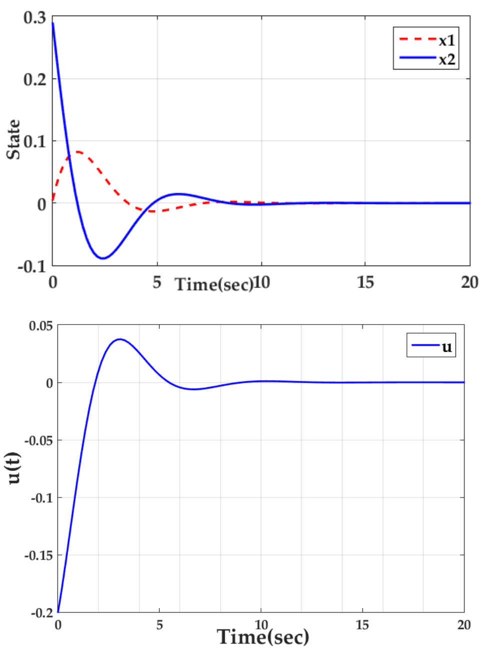


Fig. 13. Simulation E2; states and input control.

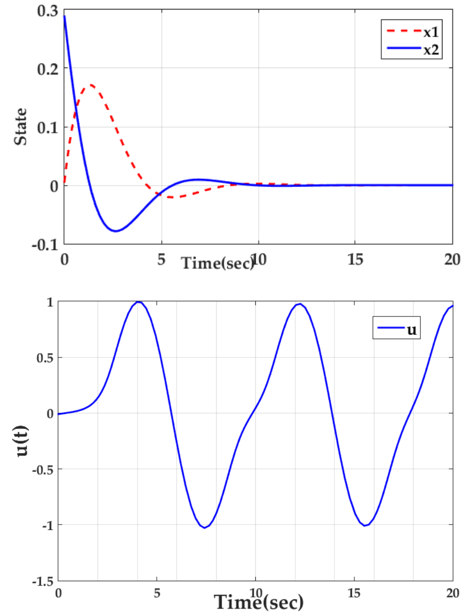


Fig. 14. Simulation results of E3; states and input control.

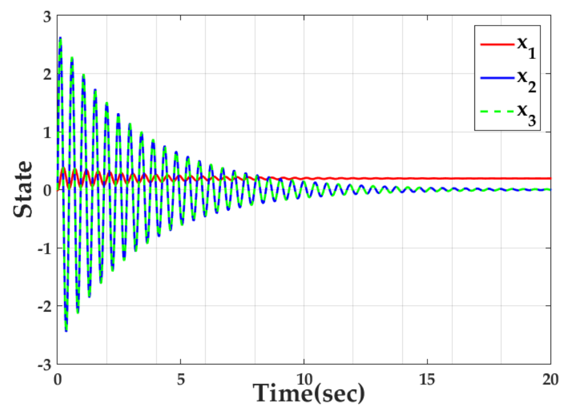


Fig. 15. Simulation results of E4; states.

VII. Conclusion

This paper discusses the concept of combining analytical and meta-heuristic approaches. This approach improve the performance of practical engineering systems using hybrid optimization techniques. Under the constraint of actuator saturation, the proposed method ensures high stability and efficiency. Among the fields of robotics and control, this is a well-known and sensitive issue. As a result, it is established that the developed technique maintains this feature and ensures the reliability of controlled systems. Particularly, the method has been proven effective in controlling robot actuators. The study focuses on the maximization of the attraction region. Furthermore, the attraction region that is studied is deducible by the defined state space polytopic regions. The main idea is to combine the generalized sector condition, with the Finsler lemma and linear annihilator in order to reduce the conservativeness. The designed outline presented in this research relied on the central idea of computing best vertices of the polytypic and determining the associated optimal Lyapunov function to find the largest attraction region. To obtain the largest domain of attraction we implemented a meta-heuristic approach, with encoding the variable of the vertices of the polytope of admissible state to be determined as particle position was presented. The meta-heuristic technique introduced in this study is powerful in problem solving, and a very efficient global search algorithm. As well as the result, the numerical example shows that using meta-heuristic leads to the biggest DA. A novel perspective on the enlarging of the DA could be performed by implementing meta optimization method for tuning the controller gains of nonlinear system with input saturation and model parameter uncertainties.

REFERENCES

[1] H. Chen, C. Wang, B. Zhang and D. Zhang, Saturated tracking control for nonholonomic mobile robots with dynamic feedback. *Transactions of the Institute of Measurement and Control*, 35(2), 105-116, 2013.

[2] D. F. Coutinho and J. G. da Silva. Estimating the region of attraction of nonlinear control systems with saturating actuators. *In 2007 American Control Conference*, (pp. 4715-4720), IEEE, 2007.

[3] J. B. Biemond and W. Michiels. Estimation of basins of attraction for controlled systems with input saturation and time-delays. *IFAC Proceedings Volumes*, 47(3), 11006-11011, 2014.

[4] H. K. Khalil. Control of nonlinear systems *Prentice Hall, New York, NY*, 2002.

[5] J. G. Da Silva and S. Tarbouriech. Antiwindup design with guaranteed regions of stability: an LMI-based approach, *IEEE Transactions on Automatic Control*, 50(1), 106-111, 2005.

[6] Y. Huang and A. Jadbabaie. Nonlinear H ∞ control: An enhanced quasi-LPV approach, *IFAC Proceedings Volumes*, 32(2), 2754-2759, 1999.

[7] G. Chesi, A. Garulli, A. Tesi and A. Vicino. Robust analysis of LFR systems through homogeneous polynomial Lyapunov functions *IEEE Transactions on Automatic Control*, 49(7), 1211-1215, 2004.

[8] L. El Ghaoui, and G. Scorletti. Control of rational systems using linear-fractional representations and linear matrix inequalities *Automatica*, 32(9), 1273-1284.,1996.

[9] D. Coutinho, A. Trofino and M. Fu. Guaranteed cost control of uncertain nonlinear systems via polynomial Lyapunov functions, *IEEE Transactions on Automatic control*, 47(9), 1575-1580, 2002.

[10] A. Trofino and T.J. M. Dezuo. LMI stability conditions for uncertain rational nonlinear systems, *International Journal of Robust and Nonlinear Control*, 24(18), 3124-3169., 2014.

[11] D. F. Coutinho, C. E. de Souza, and A. Trofino. Stability analysis of implicit polynomial systems *IEEE Transactions on Automatic Control*, 54(5), 1012-1018., 2009.

[12] D. F. Coutinho and J. G. Da Silva. Computing estimates of the region of attraction for rational control systems with saturating actuators, *IET control theory & applications*, 4(3), 315-325., 2010.

[13] S. Rozgonyi, K. Hangos and G. Szederkényi. Determining the domain of attraction of hybrid non-linear systems using maximal Lyapunov functions, *Epidemiology*, 46(1), 19-37., 2010.

[14] S. Dubljević and N. Kazantzis. A new Lyapunov design approach for nonlinear systems based on Zubov's method *Automatica*, 38(11), 1999-2007.

[15] G. Chesi, A. Garulli, A. Tesi and A. Vicino. LMI-based computation of optimal quadratic Lyapunov functions for odd polynomial systems, *International Journal of Robust and Nonlinear Control: IFAC-Affiliated Journal*, 15(1), 35-49. 2005.

[16] X. S.Yang. Engineering optimization: an introduction with metaheuristic applications, *John Wiley & Sons*

[17] U. Topcu, A. Packard, P. Seiler. Local stability analysis using simulations and sum-of-squares programming, *Automatica*, 44(10), 2669-2675, 2008.

[18] J. L.Pitarch, A. Sala, C. V.Arino. Closed-form estimates of the domain of attraction for nonlinear systems via fuzzy-polynomial models, *IEEE transactions on cybernetics*, 44(4), 526-538., 2013.

[19] F. Hamidi, M. Aloui, H. Jerbi, M. Kchaou, R. Abbassi, D. Popescu, B. E. N.Sondess and C. Dimon. Chaotic particle swarm optimisation for enlarging the domain of attraction of polynomial nonlinear systems, *Electronics*, 9(10), 1704., 2020.

[20] E. Najafi, R. Babuška and G. A.Lopes. A fast sampling method for estimating the domain of attraction, *Nonlinear dynamics*, 86, 823-834., 2016.

[21] C. K.Luk and G. Chesi. On the estimation of the domain of attraction for discrete-time switched and hybrid nonlinear systems, *International Journal of Systems Science*, 46(15), 2781-2787., 2015.

[22] H. Jerbi, B. Dabbagui, F. Hamidi, M. Aoun, Y. Bouazzi and S. B.Aoun. Computing the Domain of Attraction using Numerical Techniques, *2022 4th International Conference on Applied Automation and Industrial Diagnostics (ICAAID)*, Vol. 1, pp. 1-5, 2022.

[23] H. Jerbi. Estimations of the domains of attraction for classes of nonlinear continuous polynomial systems, *Arabian Journal for Science and Engineering*, 42(7), 2829-2837, 2017.

[24] W. J.Jemai, H. Jerbi and M. N.Abdelkrim. Nonlinear state feedback design for continuous polynomial systems, *International Journal of Control, Automation and Systems*, 9, 566-573., 2011.

[25] H. M. Jerbi, F. Hamidi, B.E. N. Sondess, S. C. Olteanu and D. Popescu. Lyapunov-based methods for maximizing the domain of attraction, *International Journal of Computers Communications & Control*, 15(5)., 2020.

[26] B. Dabbaghi, F. Hamidi, H. Jerbi and M. Aoun. Estimating and enlarging the domain of attraction for a nonlinear system with input saturation, *2023 IEEE International Workshop on Mechatronic Systems Supervision-IW-MSS*, (pp. 1-5), 2023.

[27] F. Hamidi, H. Jerbi, S. C. Olteanu and D. Popescu. An enhanced stabilizing strategy for switched nonlinear systems, *Studies in Informatics and Control*, 28(4), 391-400., 2019.

[28] D. F.Coutinho, A. S.Bazanella, A. Trofino and A. S.e Silva, (2004). An enhanced stabilizing strategy for switched nonlinear systems, *International Journal of Robust and Nonlinear Control: IFAC-Affiliated Journal*, 14(16), 1301-1326., 2004.

[29] T. Kiyama and T. Iwasaki. On the use of multi-loop circle criterion for saturating control synthesis, *Systems & Control Letters*, 41(2), 105-114., 2000.

[30] F. Hamidi, H. Jerbi, H. Alharbi, V. Leiva, D. Popescu and W. Rajhi. Metaheuristic Solution for Stability Analysis of Nonlinear Systems Using an Intelligent Algorithm with Potential Applications, *Fractal and Fractional*, 7(1), 78., 2023.

[31] J. C. Bansal, P. K. Singh and N. R.Pal. Evolutionary and swarm intelligence algorithms, *Springer*, (Vol. 779)., 2019.

[32] S. L.Tilahun and J. M. T.Ngnotchouye. Firefly algorithm for discrete optimization problems: A survey, *KSCE Journal of civil Engineering*, 21, 535-545., 2017.

[33] W. Deng, J. Xu and H. Zhao. An improved ant colony optimization algorithm based on hybrid strategies for scheduling problem, *IEEE access*, 7, 20281-20292., 2019.

[34] G. Wu, X. Shen, H. Li, H. Chen, A. Lin and P. N.Suganthan. Ensemble of differential evolution variants, *Information Sciences*, 423, 172-186., 2018.

[35] T. G. Pradeepmon, V. V.Panicker and R. Sridharan. A heuristic algorithm enhanced with probability-based incremental learning and local search for dynamic facility layout problems, *International Journal of Applied Decision Sciencess*, 11(4), 352-389., 2018.

[36] S. Mirjalili and S. Mirjalili. Biogeography-based optimisation, *Evolutionary Algorithms and Neural Networks: Theory and Applications*, 57-72., 2019.

[37] H. B.Ouyang, L. Q.Gao, S. Li, X. Y.Kong, Q. Wang and D. X.Zou. Improved harmony search algorithm: LHS, *Applied Soft Computing*, 53, 133-167, 2017.

[38] X. Li, J. Zhang and M. Yin. Animal migration optimization: an optimization algorithm inspired by animal migration behavior, *Neural computing and applications*, 24, 1867-1877, 2014.

[39] G. Singh and A. Singh. Comparative study of krill herd, firefly and cuckoo search algorithms for unimodal and multimodal optimization, *International Journal of Intelligent Systems and Applications in Engineering*, 2(3), 26-37, 2014.

[40] M. Aloui, F. Hamidi H. Jerbi, M. Omri, D. Popescu and R. Abbassi. A chaotic krill herd optimization algorithm for global numerical estimation of the attraction domain for nonlinear systems, *Mathematics*, 9(15), 1743, 2021.

[41] Z. Cui, F. Li and W. Zhang. Bat algorithm with principal component analysis, *International Journal of Machine Learning and Cybernetics*, 10, 603-622, 2019.

[42] M. Mishra, V. R. Gunturi, and D. Maity. Teaching–learning-based optimisation algorithm and its application in capturing critical slip surface in slope stability analysis, *Soft Computing*, 24(4), 2969-2982., 2020.

[43] K. Arun Vikram, C. Ratnam, V. V. K.Lakshmi, A. Sunny Kumar and R. T.Ramakanth. Application of dragonfly algorithm for optimal performance analysis of process parameters in turn-mill operations-A case study, *IOP conference series: materials science and engineering*, 24(4), 2969-2982., 2018.

[44] A. I.Zečević and D. D.Šiljak. Estimating the region of attraction for large-scale systems with uncertainties, *Automatica*, 46(2), 445-451., 2010.

[45] B. Tibken. Estimation of the domain of attraction for polynomial systems via LMIs, *Proceedings of the 39th IEEE Conference on Decision and Control (Cat. No. 00CH37187*, (Cat. No. 00CH37187) (Vol. 4, pp. 3860-3864)., 2010.

[46] U. Topcu, A. K.Packard, P. Seiler and G. J. Balas. Robust region-of-attraction estimation, *IEEE Transactions on Automatic Control*, 55(1), 137-142., 2009.

[47] M. C. de Oliveira and R. E.Skelton. Stability tests for constrained linear systems, *Springer*, p. 241-257, 2007.

[48] S. Tarbouriech, G. Garcia, J. M. G. da Silva Jr and I. Queinnec. Stability and stabilization of linear systems with saturating actuators, *Springer Science & Business Media*, 2011.

[49] J. Kennedy and R. Eberhart. Stability and stabilization of linear systems with saturating actuators, *Proceedings of ICNN'95-international conference on neural networks*, (Vol. 4, pp. 1942-1948), 1995.

[50] H. N.Wu and K. Y. Cai. Mode-independent robust stabilization for uncertain Markovian jump nonlinear systems via fuzzy control, *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, 36(3), 509-519., 2006.

[51] G. Chesi. Computing output feedback controllers to enlarge the domain of attraction in polynomial systems, *IEEE Transactions on Automatic Control*, 49(10), 1846-1853., 2004.

[52] A. Messaoudi, H. Gassara and A. El Hajjaji. Adaptive fault estimation and fault tolerant control for polynomial systems: application to electronic and mechanical systems, *Mathematical Problems in Engineering*, (1), 6342604., 2021.

[53] Y. Fu, D. Liu, J. Chen and L. He. Secretary bird optimization algorithm: a new metaheuristic for solving global optimization problems *Artificial Intelligence Review*, 57(5), 1-102, 2024.

[54] M.Aloui, F. Hamidi and H. Jerbi. Maximizing the Domain of attraction of nonlinear systems: A PSO optimization approach. *In 2021 18th International Multi-Conference on Systems, Signals & Devices (SSD)*, (pp. 375-380).

[55] F. Hamidi, H. Jerbi, W. Aggoune, M. Djemai, and M. N. Abdelkrim. Enlarging the domain of attraction in nonlinear polynomial systems. *International Journal of Computers Communications & Control*, 8(4), 538-547.

[56] T. A.Lima, D. D. S. Madeira, V. V. Viana, and R. C. Oliveira. Static output feedback stabilization of uncertain rational nonlinear systems with input saturation. *Systems & Control Letters*, 168, 105359.