Identification Problem of Source Term of A Reaction Diffusion Equation

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***Abstract*—This paper will give the numerical difference scheme with Dirichlet boundary condition， and prove stability and convergence of the difference scheme, final numerical experiment results also confirm effectiveness of the algorithm.**

Keywords- Fractional derivative; Numerical difference scheme; The gradient regularization method.

I. INTRODUCTION

Source term inversion in groundwater pollution is a class of inverse problems[7,8], and is also important field of inverse problem research. Scholars have done a large amount of work and obtained many results. Reference [2] presented a new gradient regularization algorithm to solve an inverse problem of determining source terms in one-dimension solute transport with final observations, and reference [3] proposed a implicit method to solve a class of space-time fractional order diffusion equations with variable coefficient.

However, when fractional derivative replaces second derivative in diffusion equations, there is anomalous diffusion phenomenon. In this paper, we give the numerical difference scheme in source term identification with Dirichlet boundary condition, and we prove the stability and convergence of the difference scheme, also verify the practicality and effectiveness of the algorithm through numerical experiment.

In this paper, we use difference scheme to solve the forward problem, and when solving the inverse problem, we use the gradient regularization method based on Tikhonov regularization strategy. Here, the additional information for source term identification is set as the final observations, and suppose that the source term function are only concerned with the space variable and has nothing to do with the time variable.

In fact, the solute transport model can be described by the following equation[1]

$$\frac{\partial y}{\partial t} = u(x)\frac{\partial^2 y(x,t)}{\partial x^2} - v(x)\frac{\partial y(x,t)}{\partial x} - c(x)y(x,t) + b(x,t), \quad x \in [0,L], t \in [0,T], \tag{1}$$

by introducing fractional derivative and adding initial boundary conditions[5], Eqs. (1) will be modified as the following problem

$$\frac{\partial^\alpha y}{\partial t^\alpha} = u(x)\frac{\partial^\beta y(x,t)}{\partial x^\beta} - v(x)\frac{\partial y(x,t)}{\partial x} - c(x)y(x,t) + b(x,t), \qquad x \in \Omega, t > 0, \tag{2}$$

$$Dy(x,t) = g(x,t), \qquad x \in \Omega, t > 0, \tag{3}$$

$$Ey(x,0) = f(x), \qquad x \in \Omega, \tag{4}$$

where $0 < \alpha < 1, 1 < \beta < 2$, D represents matrix operator with boundary condition, E represents matrix operator with initial condition, $y(x,t)$ and $b(x)$ represent undetermined source term function and undetermined vector function respectively.

Inverse problem of this problem is to determine unknown vector function $b(x)$ by Eqs. (2)-(4) and the additional condition below

$$y(x,t)|_{x=T} = \psi(x). \tag{5}$$

II. NUMERICAL DIFFERENCE SCHEME

Considering the following space-time reaction diffusion equations

$$\frac{\partial^\alpha y(x,t)}{\partial t^\alpha} = u(x)\frac{\partial^\beta y(x,t)}{\partial x^\beta} - v(x)\frac{\partial y(x,t)}{\partial x} - c(x)y(x,t) + b(x,t), \tag{6}$$

$$y(x,0) = f(x), \tag{7}$$

$$y(0,t) = g_1(t), y(L,t) = g_2(t). \tag{8}$$

Where $u(x) > 0, v(x) > 0, c(x) \geq 0$ are continuous functions on $[0,L]$, $g_1(t) > 0, g_2(t) > 0$ are continuous functions on $[0,T]$, $b(x,t)$ is continuous on $[0,L] \times [0,T], \alpha \in (0,1), \beta \in (1,2), \frac{\partial^\alpha y(x,t)}{\partial t^\alpha}, \frac{\partial^\beta y(x,t)}{\partial x^\beta}$ are

Caputo fractional derivative and Riemann-Liouville fractional derivative respectively[5]

$$\frac{\partial^\alpha y(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\eta)^{-\alpha}\frac{\partial y(x,\eta)}{\partial \eta}d\eta,$$

(9)

$$\frac{\partial^\beta y(x,t)}{\partial x^\beta} = \frac{1}{\Gamma(2-\beta)}\frac{\partial^2}{\partial x^2}\int_0^t \frac{y(\xi,t)}{(x-\xi)^{\beta-1}}d\xi.$$

(10)

Suppose that Eqs. (6)-(8) has a unique and smooth enough solution. $\tau = T/n$ is time step, $\Delta x = h = L/m$ is space step, $t_k = k\tau(k=0,1,2,\ldots,n)$, $x_i = ih(i=0,1,2,\ldots,m)$. For the time fractional derivative, we usually adopt following finite difference approximation

$$\frac{\partial^\alpha y(x_i,t_{k+1})}{\partial t^\alpha}$$

$$= \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{k}\frac{y(x_i,t_{j+1})-y(x_i,t_j)}{\tau}\int_{j\tau}^{(j+1)\tau}\frac{d\xi}{(t_{k+1}-\xi)^\alpha}+\tau$$

$$= \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{j=0}^{k}\frac{y(x_i,t_{k+1-j})-y(x_i,t_{k-j})}{\tau}[(j+1)^{1-\alpha}-$$

$$j^{1-\alpha}]+O(\tau),$$

(11)

$$\frac{\partial y(x_i,t_{k+1})}{\partial x} = \frac{y(x_i,t_{k+1})-y(x_{i-1},t_{k+1})}{h}+O(h).$$

(12)

For $\frac{\partial^\beta y(x,t)}{\partial x^\beta}$, by using Grünwald's improved formula [4], we have that

$$\frac{\partial^\beta y(x_i,t_{k+1})}{\partial x^\beta} = \frac{1}{h^\beta}\sum_{j=0}^{i+1}a_j y(x_i-(j-1)h,t_{k+1})+O(\tau+h),$$

(13)

where

$$a_0 = 1, a_1 = -\beta, a_j = (-1)^j\binom{\beta}{j},$$

$$\binom{\beta}{j} = \beta(\beta-1)\ldots(\beta-j+1)/(j!),\ j=1,2,3,\ldots.$$

Substituting Eqs. (11)-(13) into Eqs. (6)-(8), we obtain

$$\frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{j=0}^{k}\frac{y(x_i,t_{k+1-j})-y(x_i,t_{k-j})}{\tau}[(j+1)^{1-\alpha}-j^{1-\alpha}]$$

$$= \frac{u(x_i)}{h^\beta}\sum_{j=0}^{i+1}a_j y(x_i-(j-1)h,t_{k+1}) - c(x_i)y(x_i,t_{k+1})$$

$$-v(x_i)\frac{y(x_i,t_{k+1}) - y(x_{i-1},t_{k+1})}{h}+b(x_i,t_{k+1})+O(\tau+h).$$

Let $u_i = u(x_i), v_i = v(x_i), c_i = c(x_i), \tilde{c}_i = \tau^\alpha\Gamma(2-\alpha)\ge 0,$

$b_i^k = b(x_i,t_k),\quad \sigma_j = (j+1)^{1-\alpha}-j^{1-\alpha},\quad j=0,1,2,\ldots,n;$ $r_i =$ $\frac{u_i\tau^\alpha}{h^\beta}\Gamma(2-\alpha)\ge 0, p_i = \frac{v_i\tau^\alpha}{h}\Gamma(2-\alpha)\ge 0, g_1^k = g_1(t_k), g_2^k$ $= g_2(t_k)$, y_i^k represents numerical solution of Eqs. (6)-(8), then we have

$$\sum_{j=0}^{k}\sigma_j(y_i^{k+1-j}-y_i^{k-j}) = -p_i(y_i^{k+1}-y_{i-1}^{k+1})+r_i\sum_{j=0}^{i+1}a_j y_{i-j+1}^{k+1}$$

$$-\tilde{c}_i y_i^{k+1}+\tilde{c}_i\tilde{b}_i^{k+1}.$$

(14)

Since local truncation error is $O(\tau+h)$, thus the difference scheme is consistent[10]. Eqs. (14) will be repalced by

When $k=0$,

$$-r_i y_{i+1}^1+(1+p_i-r_i a_1+\tilde{c}_i)y_i^1-(p_i+r_i a_2)y_{i-1}^1-r_i\sum_{j=3}^{i+1}a_j y_{i-j+1}^1$$

$$= y_i^0+\tilde{c}_i\tilde{b}_i,$$

(15)

when $k>0$,

$$-r_i y_{i+1}^{k+1}+(1+p_i-r_i a_1+\tilde{c}_i)y_i^{k+1}-(p_i+r_i a_2)y_{i-1}^{k+1}-r_i\sum_{j=3}^{i+1}a_j y_{i-j+1}^{k+1}$$

$$= y_i^k-\sum_{j=0}^{k}\sigma_j(y_i^{k+1-i}-y_i^{k-j}) = (2-2^{1-\alpha})y_i^k+\sigma_k y_i^0+\tilde{c}_i\tilde{b}_i^{k+1}-$$

$$\sum_{j=1}^{k-1}y_i^{k-j}[2(j+1)^{1-\alpha}-(j+2)^{1-\alpha}-j^{1-\alpha}]$$

$$= d_1 y_i^k-\sum_{j=1}^{k-1}y_i^{k-j}d_{j+1}+\sigma_k y_i^0+\tilde{c}_i\tilde{b}_i^{k+1},$$

(16)

where $d_j = \sigma_{j-1} - \sigma_j$, $i = 1,2,\cdots,m-1; k = 1,2,\cdots,n-1.$

III. STABILITY AND CONVERGENCE OF THE DIFFERENCE SCHEME

Lemma 2.1 For arbitrary real number a,b,c,d, we have

$$-|a|+|b|-|c|-|d| \le |-a+b-c-d|.$$

Proof. From $|b| = |-a+b-c-d+a+c+d|$

$$\le |-a+b-c-d|+|a|+|c|+|d|,$$

we obtain $-|a|+|b|-|c|-|d| \le |-a+b-c-d|$.

Lemma 2.2 (1) $a_j > 0 (j \ge 2)$. (2) For any positive integer N, we have $\sum_{j=0}^{N} a_j < 0.$

Proof. (1) Note that $a_j = (-1)^j \binom{\beta}{j}$ and $1 < \beta < 2,$ we know that $a_j > 0 (j \ge 2).$

(2) Since $(1+x)^\beta = \sum_{j=0}^{\infty} \binom{\beta}{j} x^j, x \in [-1,1],$ let $x = -1,$ then $\sum_{j=0}^{\infty} a_j = 0.$ From Lemma 2.2(1), we have that

$$\sum_{j=0}^{N} a_j = -\sum_{j=N+1}^{\infty} a_j < 0.$$

Lemma2.3 (1)$\sum_{j=1}^{n} d_j = 1 - \sigma_n$; (2)$d_j > 0, \sigma_{j-1} > \sigma_j$.

Proof. (1)From $d_j = \sigma_{j-1} - \sigma_j, \sigma_j = (j+1)^{1-\alpha} - j^{1-\alpha},$ we easily know that $\sum_{j=1}^{n} d_j = 1 - \sigma_n.$

(2) Let $h(x) = (x+1)^{1-\alpha} - x^{1-\alpha} (x \ge 1),$ then $h'(x) = (1-\alpha)[(x+1)^{-\alpha} - x^{-\alpha}] < 0,$ so $h(x)$ is decreasing function, $d_j = \sigma_{j-1} - \sigma_j = h(j-1) - h(j) > 0.$ Therefore, we have that $d_j > 0, \sigma_{j-1} > \sigma_j$.

A. Stability of the difference scheme

Theorem 2.1 Implicit difference schemes defined by Eqs. (15)-(16) are unconditionally steady for initial value[10].

Proof. Suppose that $\tilde{y}_i^k, y_i^k$ represent solutions of Eqs. (15)-(16) for initial value $f_1(x), f_2(x)$ respectively, and b_i^k is accurate vale, then error $\varepsilon_i^k = \tilde{y}_i^k - y_i^k$ satisfies

When $k = 0,$

$$-r_i \varepsilon_{i+1}^1 + (1 + p_i - r_i a_1 + c_i)\varepsilon_i^1 - (p_i + r_i a_2)\varepsilon_{i-1}^1$$
$$-r_i \sum_{j=3}^{i+1} a_j \varepsilon_{i-j+1}^1 = \varepsilon_i^0,$$

when $k > 0,$

$$-r_i \varepsilon_{i+1}^{k+1} + (1 + p_i - r_i a_1 + c_i)\varepsilon_i^{k+1} - (p_i + r_i a_2)\varepsilon_{i-1}^{k+1}$$
$$-r_i \sum_{j=3}^{i+1} a_j \varepsilon_{i-j+1}^{k+1} = d_1 \varepsilon_i^k + \sum_{j=1}^{k-1} d_{j+1} \varepsilon_i^{k-j} + \sigma_k \varepsilon_i^0.$$

Let $E^k = (\varepsilon_1^k, \varepsilon_2^k, \cdots, \varepsilon_{m-1}^k)'$, we prove $\|E^k\|_\infty \le \|E^0\|_\infty$ with mathematical induction in the following.

When $k = 1,$ let $|\varepsilon_l^1| = \max_{1 \le i \le m-1} |\varepsilon_i^1|$. Note that $r_l > 0,$ $p_l > 0,$ $a_0 = 1, a_1 = -\beta < 0,$ we have from Lemma 2.2 that

$$\|E^1\|_\infty = |\varepsilon_l^1| \le |\varepsilon_l^1| + p_l(|\varepsilon_l^1| - |\varepsilon_{l-1}^1|) - r_l \sum_{j=0}^{l+1} a_j |\varepsilon_l^1|$$
$$\le -r_l |\varepsilon_{l+1}^1| + (1 + p_l - r_l a_1 + c_i)|\varepsilon_l^1|$$
$$-(p_l + r_l a_2)|\varepsilon_{l-1}^1| - r_l \sum_{j=3}^{l+1} a_j |\varepsilon_{l-j+1}^1|.$$

Note that $p_l + r_l a_2 > 0, r_l \sum_{j=3}^{l+1} a_j > 0, 1 + p_l - r_l a_1 + c_i > 0,$ and from Lemma 2.1, we further obtain that

$$|\varepsilon_l^1| \le |-r_l \varepsilon_{l+1}^1 + (1 + p_l - r_l a_1 + c_i)\varepsilon_l^1 - (p_l + r_l a_2)\varepsilon_{l-1}^1$$
$$-r_l \sum_{j=3}^{l+1} a_j \varepsilon_{l-j+1}^1| = |\varepsilon_l^0| \le \|E^0\|_\infty.$$

Thus $\|E^1\|_\infty = |\varepsilon_l^1| \le \|E^0\|_\infty$.

Assume that we always have $\|E^k\|_\infty \le \|E^0\|_\infty$ when $k \le s,$ let $|\varepsilon_l^{s+1}| = \max_{1 \le i \le m-1} |\varepsilon_i^{s+1}|,$ then when $k = s+1,$ we have

$$|\varepsilon_l^{s+1}| \le -r_l |\varepsilon_{l+1}^{s+1}| + (1 + p_l - r_l a_1 + c_i)|\varepsilon_l^{s+1}|$$
$$-(p_l + r_l a_2)|\varepsilon_{l-1}^{s+1}| - r_l \sum_{j=3}^{l+1} a_j |\varepsilon_{l-j+1}^{s+1}|$$
$$\le |-r_l \varepsilon_{l+1}^{s+1} + (1 + p_l - r_l a_1 + c_i)\varepsilon_l^{s+1}$$

$$-(p_l + r_l a_2)\varepsilon_{l-1}^{s+1} - r_l \sum_{j=3}^{l+1} a_j \varepsilon_{l-j+1}^{s+1} |$$

$$= | d_1 \varepsilon_l^s + \sum_{j=1}^{s-1} d_{j+1} \varepsilon_l^{s-j} + \sigma_s \varepsilon_l^0 |$$

$$\le d_1 \| E^s \|_\infty + \sum_{j=1}^{s-1} d_{j+1} \| E^{s-j} \|_\infty + \sigma_s \| E^0 \|_\infty$$

$$\le (d_1 + \sum_{j=1}^{s-1} d_{j+1} + \sigma_s) \| E^0 \|_\infty$$

$$= (1 - \sigma_s + \sigma_s) \| E^0 \|_\infty = \| E^0 \|_\infty .$$

Consequently, the desired result follows.

B. Convergence of the difference scheme

Suppose that $y(x_i, t_k)$ is exact solution of the differential equation at (x_i, t_k). Let $e_i^k = y(x_i, t_k) - y_i^k$, $e^k = (e_1^k, e_2^k, \cdots, e_{m-1}^k)'$, then $y_i^k = y(x_i, t_k) - e_i^k$, substituting it into Eqs. (15)-(16), and note $e^0 = 0$, we have that

When $k = 0$,

$$-r_i e_{i+1}^1 + (1 + p_i - r_i a_1 + \tilde{c}_i) e_i^1 - (p_i + r_i a_2) e_{i-1}^1$$

$$-r_i \sum_{j=3}^{i+1} a_j e_{i-j+1}^1 = R_i^1,$$

when $k > 0$,

$$-r_i e_{i+1}^{k+1} + (1 + p_i - r_i a_1 + \tilde{c}_i) e_i^{k+1} - (p_i + r_i a_2) e_{i-1}^{k+1}$$

$$-r_i \sum_{j=3}^{i+1} a_j e_{i-j+1}^{k+1} = d_1 e_i^k + \sum_{j=1}^{k-1} d_{j+1} e_i^{k-j} + R_i^{k+1},$$

where $| R_i^k | \le \lambda(\tau^{1+\alpha} + \tau^\alpha h), \lambda$ is a positive constant and it has nothing to do with h, τ.

Theorem 2.2 There is a constant $\lambda > 0$ such that

$$\| e^k \|_\infty \le \sigma_{k-1}^{-1} \lambda(\tau^{1+\alpha} + \tau^\alpha h), k = 1, 2, \cdots, n,$$

where λ has nothing to do with h, τ, and $\| e^k \|_\infty = \max_{1 \le i \le m-1} | e_i^k |$.

Proof. When $k = 1$, let $| e_l^1 | = \max_{1 \le i \le m-1} | e_i^1 |$, then $\| e^1 \|_\infty = | e_l^1 |$, we have from Lemma 2.1 that

$$| e_l^1 | \le -r_l | e_{l+1}^1 | + (1 + p_l - r_l a_1 + \tilde{c}_i) | e_l^1 |$$

$$-(p_l + r_l a_2) | e_{l-1}^1 | - r_l \sum_{j=3}^{l+1} a_j | e_{l-j+1}^1 |$$

$$\le | -r_l e_{l+1}^1 + (1 + p_l - r_l a_1 + \tilde{c}_i) e_l^1 - (p_l + r_l a_2) e_{l-1}^1$$

$$-r_l \sum_{j=3}^{l+1} a_j e_{l-j+1}^1 | = | R_l^1 |$$

$$\le \lambda(\tau^{1+\alpha} + \tau^\alpha h) = \sigma_0^{-1} \lambda(\tau^{1+\alpha} + \tau^\alpha h).$$

Assume that $\| e^s \|_\infty \le \sigma_{s-1}^{-1} \lambda(\tau^{1+\alpha} + \tau^\alpha h)$ when $k \le s$, let $| e_l^{s+1} | = \max_{1 \le i \le m-1} | e_i^{s+1} |$, then when $k = s+1$, we have that

$$\| e^{s+1} \|_\infty = | e_l^{s+1} |$$

$$\le d_1 \| e^s \|_\infty + \sum_{j=1}^{s-1} d_{j+1} \| e^{s-j} \|_\infty + \lambda(\tau^{1+\alpha} + \tau^\alpha h)$$

$$\le (1 + d_1 \sigma_{s-1}^{-1} + d_2 \sigma_{s-2}^{-1} + \cdots + d_s \sigma_0^{-1}) \lambda(\tau^{1+\alpha} + \tau^\alpha h),$$

as in Lemma 2.3, we have found that $\sigma_j^{-1} \le \sigma_s^{-1} (j \le s)$, so we futher obtain

$$\| e^{s+1} \|_\infty \le \sigma_s^{-1} (\sum_{i=0}^{s-1} d_{i+1} + \sigma_s) \lambda(\tau^{1+\alpha} + \tau^\alpha h)$$

$$= \sigma_s^{-1} \lambda(\tau^{1+\alpha} + \tau^\alpha h).$$

The desired result follows.

Since $\lim_{k \to \infty} \frac{\sigma_k^{-1}}{k^\alpha} = \lim_{k \to \infty} \frac{1}{k^\alpha \sigma_k} = \lim_{k \to \infty} \frac{1}{k[(1 + 1/k)^{1-\alpha} - 1]} = \frac{1}{1-\alpha}$, thus there is a constant $\gamma > 0$ such that

$$\| E^k \|_\infty \le k^\alpha \gamma(\tau^{1+\alpha} + \tau^\alpha h) = (k\tau)^\alpha \gamma(\tau + h).$$

Consequently, we can obtain the following result when $k\tau \le T$.

Theorem 2.3 Suppose that $y(x_i, t_k)$ denotes exact solution at (x_i, t_k), y_i^k is numerical solution of implicit difference scheme, then there exists a constant $\tilde{\gamma} = T^\alpha \gamma > 0$ such that

$$| y(x_i, t_k) - y_i^k | \le \tilde{\gamma}(\tau + h), \ 1 \le i \le m, 1 \le k \le n.$$

IV. SOURCE TERM INVERSION

The inverse problem, which is composed of Eqs. (2)-(5), is to solve nonlinear operator equation

$$A[b(x)] = \psi(x).$$

Suppose that $y(b(x);x,t)$ denotes solution of Eqs. (2)-(4) for $b(x)$, $b_0(x)$ denotes a function near $b^*(x)$, where $b_0(x)=\sum_{i=1}^{n}k_i^0\psi_i(x)=K_0^{\mathrm{T}}\Psi(x)$, and $b^*(x)$ denotes exact solution of Eqs. (2)-(4), $b(x)\in K$, $\psi_1(x),\psi_2(x),\cdots$, are a group of primary functions on K, then a tiny disturbing quantity of $b_0(x)$ is

$$\delta b_0(x)=\sum_{i=1}^{n}\delta k_i^0\psi_i(x)=\delta K_0^{\mathrm{T}}\Psi(x),$$

(17)

where $\Psi(x)=(\psi_1(x),\psi_2(x),\cdots,\psi_n(x))^T$, $K^T=(k_1,k_2,\cdots,k_n)\in R^n$.

Assume that $y(b_1(x);x,t)$ denotes the solution of initial boundary value problem for $b_1(x)$, where $b_1(x)=b_0(x)+\delta b_0(x)$, using Taylor formula, then we have[6]

$$y(b_0(x)+\delta b_0(x);x,t)$$

$$=y(b_0(x);x,t)+\nabla_{K_0}^T y(b_0(x);x,t)\cdot\delta K_0+o(\|\delta b_0(x)\|),$$

by using the Tikhonov regularization method, solving $b(x)$ is converted into δK_0, and δK_0 can be determined by local minimum of the following function[9]

$$G[\delta K_0]=\|y(b_0(x)+\delta b_0(x);x,t)-y(x,T)\|^2_{L^2(\partial\Omega'\times[0,T])}$$

$$+\alpha S[\delta K_0]=\|y(b_0(x);x,t)-y(x,T)+$$

$$\nabla_{K_0}^T y(b_0(x);x,t)\cdot\delta K_0\|^2_{L^2(\partial\Omega'\times[0,T])}+\alpha S[\delta K_0],$$

(18)

where $x\in\partial\Omega'\subset\partial\Omega$, α denotes regularization coefficient, $S[\delta K_0]$ denotes steady functional of δK_0.

Assume that there are discrete points $x_m(m=1,2,\cdots,M)$ on $\partial\Omega'$, $S[\delta K_0]=\delta K_0^T\delta K_0$, then

$$G[\delta K_0]=\delta K_0^T A^T A\delta K_0+2\delta K_0^T A^T(P-Q)$$

$$+(P-Q)^T(P-Q)+\alpha\delta K_0^T\delta K_0.$$

(19)

It is easy to prove that solving the local minimum values of Eqs. (19) is equivalent to slove $(A^TA+\alpha I)\delta K_0=A^T(Q-P)$, so we have

$$\delta K_0=(A^TA+\alpha I)^{-1}A^T(Q-P),$$

(20)

where

$$P=\begin{bmatrix} y(b_0(x);x_1,T)\\ y(b_0(x);x_2,T)\\ \vdots\\ y(b_0(x);x_M,T)\end{bmatrix},\qquad Q=\begin{bmatrix} y_T(x_1)\\ y_T(x_2)\\ \vdots\\ y_T(x_M)\end{bmatrix},$$

$$A=(a_{ij})_{M\times N},\ a_{ij}=\frac{\partial}{\partial k_j}y(b_0(x);x_i,T).$$

Substituting Eqs. (20) into Eqs. (17), we can obtain $\delta b_0(x)$ and a new approximation of the exact solution, $b_1(x)=b_0(x)+\delta b_0(x)$.

Repeating the above, until satisfies the precision requirement.

V. NUMERICAL EXPERIMENTS

In order to verify the effectiveness of the algorithm in the source term identification, we do the following numerical experiment[7]. For simplicity, we set part variables as follows

$$c(x)=0.05,u(x)=292,v(x)=365,L=4000,T=11,$$
$$m=20,n=11,\Delta=[0.01,0.01,0.01].$$

Where Δ is the increment of K when computing matrix A from Eqs. (20). Moreover, we always take polynomial function space as primary function space in the following computation, and setting initial boundary condition as follows

$$y_T(x)=0.057x+45.6, 0\le x\le 4000,$$

$$g_1(t)=0.724t^2+45.6, 0\le t\le 11,$$

$$g_2(t)=2.2t^2+273.4, 0\le t\le 11.$$

Let $b(x)=1-x$ in the Eqs. (6)-(8), and substituting initial boundary condition, we can obtain $y(X,T)$ by sloving forward problem. And as the additional data, we can do inversion calculation by using the above algorithm. Let initial iteration vector $K_0=[1,1,1]$, and iterative termination condition $\delta b(x)<1e-4$, then we obtain inversion results under different regularization coefficient(see TABLE Ⅰ), let $theta=1e-3$, we get inversion results under different initial value(see TABLE Ⅱ).

TABLE Ⅰ. THE INVERSION RESULTS UNDER DIFFERENT REGULARIZATION COEFFICIENT

θ	Iteration times	Results

1e-1	4	1.0000 -1.0000 0
1e-2	3	1.0000 -1.0000 0
1e-3	3	1.0000 -1.0000 0
1e-4	3	1.0000 -1.0000 0

TABLE Ⅱ. THE INVERSION RESULTS UNDER DIFFERENT INITIAL VALUE

K_0	Iteration times	Results
-100 -100 -100	4	1.0000 -1.0000 0
-10 -10 -10	3	1.0000 -1.0000 0
1 1 1	3	1.0000 -1.0000 0
10 10 10	3	1.0000 -1.0000 0
100 100 100	4	1.0000 -1.0000 0

To better simulate the errors generated by actual data, and verify the effectiveness of the algorithm, we choose the disturbance error $V^{\rho} = V(1+\xi\rho)$, where $\xi \in [-1,1]$, and $\rho > 0$ is error level.

According to the above algorithm, we do 8 times numerical experiments, and obtain the inversion results under different error level ρ (see TABLE Ⅲ). Besides, the comparison of inversion results and exact solution can see Figure 1 when ρ =0.01.

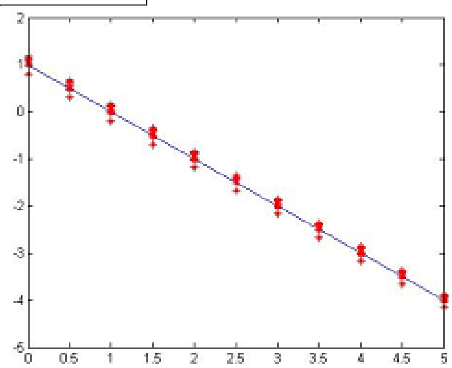

Figure 1. The comparison of inversion results and exact solutions

TABLE Ⅲ. THE INVERSION RESULTS UNDER DIFFERENT ERROR LEVEL

Times	ρ =0.01	ρ =0.05	ρ =0.1
1	0.9941 -0.9997 -0.0000	1.4988 -1.0243 0.0000	0.3838 -0.9700 -0.0000
2	1.1639 -1.0080 0.0000	0.3221 -0.9670 -0.0000	2.3115 -1.0639 0.0000
3	1.1098 -1.0053 0.0000	0.8026 -0.9903 -0.0000	-1.0527 -0.9000 -0.0000
4	0.9818 -0.9991 -0.0000	1.9119 -1.0444 0.0000	-0.5123 -0.9264 -0.0000
5	0.7984 -0.9902 -0.0000	1.8730 -1.0425 0.0000	-0.2448 -0.9394 -0.0000
6	1.1346 -1.0066 0.0000	0.8121 -0.9909 -0.0000	-0.2617 -0.9386 -0.0000
7	0.9768 -0.9989 -0.0000	1.8243 -1.0401 0.0000	1.4347 -1.0212 0.0000
8	1.0483 -1.0024 0.0000	0.0742 -0.9549 -0.0000	0.0459 -0.9535 -0.0000
mean value	0.9941 -1.0013 0.0000	1.1399 -1.0067 0.0000	0.2631 -0.9641 -0.0000

Through the above numerical experiment, we find that the inversion results and exact solution are almost the same, and this shows that the above algorithm is feasible and very effective.

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