A Mixed Finite Element Method for Elasticity Problem

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Abstract—**This paper describes a numerical solution for plane elasticity problem. It includes algorithms for discretization by mixed finite element methods. The discrete scheme allows the utilization of Brezzi - Douglas - Marini element (BDM$_1$) for the stress tensor and piecewise constant elements for the displacement. The numerical results are compared with some previously published works or with others coming from commercial code like ABAQUS.**

keywords—Elasticity problem; Mixed Finite element method; BDM$_1$ approximation; ABAQUS

I. INTRODUCTION

Mixed finite element methods for linear elasticity are based on approximations of a stress- displacement system derived from the Hellinger-Reissner variational principle [7], in which both displacements and stresses were approximated simultaneously.

The mathematical analysis and applications of mixed finite element methods have been widely developed since the seventies. A general analysis for this kind of methods was first developed by Brezzi [8]. We also have to mention the papers by Babuska [14] and by Crouzeix and Raviart [15] which, although for particular problems, introduced some of the fundamental ideas for the analysis of mixed methods.

We also refer the reader to [16][17], where general results were obtained, and to the books [6][18][19].

Many mixed finite element methods have been developed for plane elasticity, and generally speaking, they can be grouped into two categories: methods that enforce the symmetry of the stress weakly, and methods that enforce the symmetry exactly (strongly). In the former category, the stress tensor is not necessarily symmetric, but rather orthogonal to anti-symmetric tensors up to certain moments. Weakly imposed stress symmetry methods also introduce a new variable into the formulation that approximates the anti-symmetric part of the gradient of u; see for example [2][3].On other hand, exactly symmetric stress methods have been much more difficult to construct. The first class of

inf_sup stable methods was the so called composite elements [4][5].

Section II presents the model problem used in this paper. The discretization by mixed finite elements described is in section III. Numerical experiments carried out within the framework of this publication and their comparisons with other results are shown in section IV.

II. GOVERNING EQUATIONS

The equilibrium equations and boundary conditions are

$$\nabla . \sigma \; + \; f \; = \; 0 \; in \; \Omega \qquad (1)$$
$$\sigma . n \; = \; \bar{t} \; on \; \Gamma_t \qquad (2)$$
$$\sigma . n \; = \; 0 \; on \; \Gamma_{c+} \qquad (3)$$
$$\sigma . n \; = \; 0 \; on \; \Gamma_{c-} \qquad (4)$$

Where n is the unit outward normal. In the above, σ is the Cauchy stress, and f is the body force per unit volume.

The constitutive relation is given by Hooke's law:
$$\sigma \; = \; C : \varepsilon \qquad (5)$$

Where C is the Hooke tensor, C is assumed here to have constant coefficients. Its inverse (compliance tensor) will be denoted by E. Hence

$$\sigma \; = \; C : \varepsilon \; \Leftrightarrow \; \varepsilon = \; E : \sigma. \qquad (6)$$

We consider small strains and displacements. The kinematics equations therefore consist of the strain-displacement relation

$$\varepsilon = \varepsilon(u) = \; \nabla_S u \qquad (7)$$

Where $\nabla_s u = \frac{1}{2}(\nabla u + \nabla u^T)$ is the symmetric part of the gradient operator, and the boundary condition
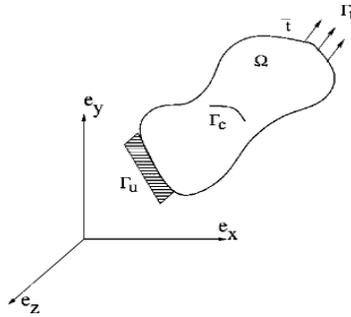
$$u \; = \; \bar{u} \; on \; \Gamma_u \qquad (8)$$

Fig. 1.   Body With Internal Boundary Subjected To Loads.

We set

$$H(div, \Omega) = \{\sigma | \sigma \in (L^2(\Omega))^4 ;$$

$$\sigma_{ij} = \sigma_{ji} \quad \forall i, j ; div\sigma \in (L^2(\Omega))^2\}. \quad (9)$$

$$H_g(div, \Omega) = \{\tau \in H(div, \Omega); \tau.n = g \text{ on } \Gamma_t\} \quad (10)$$

$$L^2_{dis}(\Omega) = \{u \in (L^2(\Omega))^2; u \text{ discontinuous on } \Gamma_c\} \quad (11)$$

Then the standard weak formulation of the equilibrium equations is the following:

Find $\sigma \in H_{\bar{t}}(div, \Omega)$ and $u \in L^2_{dis}(\Omega)$ such that:

$$\int_\Omega (E : \sigma): \tau \, dx + \int_\Omega u. div \, \tau \, dx$$
$$= \int_{\Gamma_u} \bar{u} \, \tau.n \, d\Gamma, \quad (12)$$

$$\forall \tau \in H_0(div, \Omega)$$
$$\int_\Omega v. div\sigma \, dx + \int_\Omega f.v \, dx = 0 \ \forall v \in L^2_{dis}(\Omega). \quad (13)$$

One can see that (12)-(13) practically coincide with the variational formulation of the Hellinger-Reissner principle. The use of this principle in the framework of finite elements can be traced Back to the pioneering work of Herrmann [9] and Hellan [10]. The interest in using the stress field σ as an independent variable is questionable in as simple a case as the present one, but it is clear in more general and more complicated problems involving nonlinearities, plasticity, and so on.

Let the bilinear forms a and b, and the linear forms l and s such that:

$$a(\sigma, \tau) = \int_\Omega (E : \sigma): \tau \, dx \quad (14)$$

$$b(\sigma, u) = \int_\Omega u. div \, \sigma \, dx \quad (15)$$

$$l(v) = -\int_\Omega f.v \, dx \quad (16)$$

$$s(\tau) = \int_{\Gamma_u} \bar{u} \, \tau.n \, d\Gamma, \qquad for \ all \ \tau \in H(div, \Omega). \quad (17)$$

The underlying weak formulation (12)-(13) may be restated as:

Find $\sigma \in H_{\bar{t}}$ and $u \in L^2_{dis}(\Omega)$ such that:

$$a(\sigma, \tau) + b(\tau, u) = s(\tau), for \ all \ \tau \in H_0(div, \Omega) \quad (18)$$

$$b(\sigma, v) = l(v), for \ all \ v \in L^2_{dis}(\Omega). \quad (19)$$

THEOREM 1. Let E and Ψ be real Hilbert spaces, $a(\xi_1, \xi_1)$ a bilinear form on E × E, and $b(\xi, \psi)$ a bilinear form an E × Ψ. Set

$$K = \{\xi | \xi \in E, b(\xi, \psi) = 0 \ \forall \psi \in \Psi\}, \quad (20)$$

And assume that:

$$\exists \alpha > 0, \quad such \ that \ a(\xi, \xi) \geq \alpha \parallel \xi \parallel^2_E , \forall \xi \in K \quad (21)$$

$$\exists \beta > 0, \quad such \ that \ \underset{\xi \epsilon E - \{0\}}{SUP} \frac{b(\xi, \psi)}{\|\xi\|_E} \geq \beta \|\xi\|_\Psi ,$$
$$\forall \psi \in \Psi \quad (22)$$

Then for every $l_1 \in E'$ and $l_2 \in \Psi'$ there exist a unique solution $(\bar{\bar{\xi}}, \bar{\psi})$ of the problem

$$a(\bar{\bar{\xi}}, \xi) + b(\xi, \bar{\psi}) = \langle l_1, \xi \rangle, for \ all \ \xi \in E \quad (23)$$

$$b(\bar{\bar{\xi}}, \psi) = \langle l_2, \psi \rangle, for \ all \ \psi \in \Psi. \quad (24)$$

REMARK 1. If problem (21)-(22) has a unique solution for every $l_1 \in E'$ and $l_2 \in E'$, then (20) holds and the bilinear form $a(\xi_1, \xi_2)$ restricted to K, is nonsingular (in the sense that it induces an isomorphism from K onto K'). Clearly if one assumes that a (ξ_1, ξ_2) is symmetric and positive semi definite, then (21) and (22) are necessary and sufficient for the existence and uniqueness of the solution of (23)-(24).

REMARK 2. It is clear that if a (ξ_1, ξ_2) is symmetric; the solution $(\bar{\bar{\xi}}, \bar{\psi})$ of (23)-(24) minimizes the functional

$$J(\xi) = \frac{1}{2} a(\xi, \xi) - \langle l_1, \xi \rangle \quad (25)$$

On the subspace of E,

$$K(l_2) = \{\xi \mid \xi \in E, b(\xi, \psi) = \langle l_2, \psi \rangle \ \forall \psi \in \Psi\} \quad (26)$$

And the formulation (23)-(24) corresponds to the introduction in (25)-(26) of the Lagrange multiplier $\bar{\xi}$.

III.   MIXED FINITE ELEMENT APPROXIMATION

Let T_h; $h > 0$, be a family of rectangulations of Ω.

The edges of elements will be denoted $e_i (i=1, 2, 3 \ or \ i=1, 2, 3, 4)$ in the two-dimensional case. Let us deal first with the abstract framework (23)-(24). Assume that we are

given two sequences $\{E_h\}_{h>0}$ and $\{\Psi_h\}_{h>0}$ of subspaces E and Ψ, respectively.

We set

$$K_h = \{\xi_h|\xi_h \in E_h, b(\xi_h,\psi_h) = 0 \; \forall \psi_h \in \Psi_h\}. \quad (27)$$

We have the following approximation theorem

THEOREM 2. Assume that

$$\exists \alpha_h > 0, \;\; such \; that \; a(\xi,\xi) \geq \alpha_h \|\xi\|_E^2, \forall \xi \in K_h \quad (28)$$

$$\exists \beta_h > 0, \;\; such \; that \; \underset{\xi \in E_h - \{0\}}{SUP} \frac{b(\xi,\psi)}{\|\xi\|_E} \geq \beta_h \|\psi\|_\Psi \quad (29)$$

$$\forall \psi \in \Psi_h$$

Then for every $l_1 \in E'$ and $l_2 \in \Psi'$, and for every $h > 0$, the discrete problem

$$a(\bar{\bar{\xi}}_h,\xi) + b(\xi,\bar{\psi}_h) = \langle l_1, \xi \rangle, for \; all \; \xi \in E_h, \quad (30)$$

$$b(\bar{\bar{\xi}}_h,\psi) = \langle l_2, \psi \rangle, for \; all \; \psi \in \Psi_h \quad (31)$$

Has a unique solution. Moreover, there exists a constant $\gamma_h(\alpha_h,\beta_h) > 0$ such that

$$\|\bar{\xi} - \bar{\bar{\xi}}_h\|_E + \|\bar{\psi} - \bar{\psi}_h\|_\Psi \leq \gamma_h(\underset{\xi_h \in E_h}{inf} \|\bar{\xi} - \xi_h\|_E + \underset{\psi_h \in \Psi_h}{inf} \|\bar{\psi} - \psi_h\|_\Psi). \quad (32)$$

The dependence of γ_h on α_h and β_h can be easy traced [8]. Clearly if (21) and (22) hold with constants $\bar{\alpha}$ and $\bar{\beta}$ independent of h, then (32) holds with a constant $\bar{\gamma}$ independent of h.

We define in general, for m integer ≥ 0,

$$H^m(\Omega) = \{v|D^\alpha v \in L^2(\Omega), \forall |\alpha| \leq m\} \quad (33)$$

Where

$$D^\alpha v = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}, |\alpha| = \alpha_1 + \dots + \alpha_n \quad (34)$$

These derivatives being taken in the sense of distributions. On this space, we shall use the semi-norm

$$|v|_{m,\Omega}^2 = \sum_{|\alpha|=m} |D^\alpha v|_{L^2(\Omega)}^2 \quad (35)$$

and the norm

$$\|v\|_m^2 = \sum_{K \leq m} |v|_{k,\Omega}^2. \quad (36)$$

We are now ready for the error estimates.

THEOREM 3. If (σ, u) is the solution of (12)-(13) and (σ_h, u_h) is the solution of (30)-(31), there exist a constant C > 0 such that:

$$\|\sigma - \sigma_h\|_0 + \|u - u_h\|_0 \leq Ch^2(\|\sigma\|_2 + \|u\|_3). \quad (37)$$

Discretization of the mixed formulations, for linear elliptic operators, many examples of successful discretization of (12)-(13) are known. The first ones were introduced by Raviart and Thomas in [11] and then re-elaborated and extended to more general cases by Nedelec [12]. Other families of possible discretization were introduced years later by Brezzi, Douglas, and Marini [1][13].

To give a more precise definition of our mixed finite element approximation we shall need a few definitions. Let us define on an element K.

P_k: the space of polynomials of degree $\leq$ k.

We shall also need polynomial spaces on the edges of the elements

$$R_k(\partial K) = \{\phi|\phi \in L^2(\partial K), \phi|_{e_i} \in P_k(e_i), \forall e_i \in \partial K\}. \quad (38)$$

In the two-dimensional, for the triangular elements we have

$$BDM_k(K) = (P_k(K))^2, (k \geq 1) \quad (39)$$

$$BDFM_k(K) = \{q \in (P_k(K))^2 \mid q.n|_{\partial K} \in R_{k-1}(\partial K)\}, (k \geq 1) \quad (40)$$

$$RT_k(K) = (P_k(K))^2 \oplus x(P_k(K)), (k \geq 1). \quad (41)$$

Restricting $q \in BDM_k(K)$ to have a normal trace in $R_{k-1}(\partial K)$ yields a space larger than $RT_k(K)$, but having essentially the same properties, that we denote $BDFM_k(K)$.

The dimension of BDM_k is thus

$$\dim BDM_k = \begin{cases} (k+1)(k+2) & for \; n = 2 \\ \frac{1}{2}(k+1)(k+2)(k+3) & for \; n = 3. \end{cases} \quad (42)$$

For the triangular case we thus have the following inclusions between the spaces just defined

$$RT_0 \subset BDFM_1 \subset BDM_1 \subset RT_1 \subset BDFM_2 \subset BDM_2 \subset RT_2 \quad (43)$$

We consider the space obtained basically from the space of Brezzi-Douglas-Marini.

$$BDM_1(K) = (P_1(K))^2. \quad (44)$$

We have

$$div(BDM_1(K)) = P_0(K). \quad (45)$$
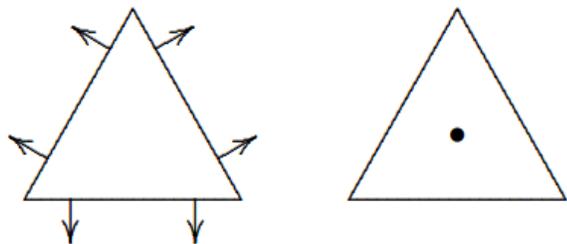
$$\dim BDM_1 = 6 \quad (46)$$

Fig. 2.   . Brezzi - Douglas - Marini element, the element diagrams for the stress and displacement elements.

The discrete scheme allows the utilization of BDM$_1$ for the stress tensor and piecewise constant elements for the displacement.

We define, for choice of BDM$_1$ (K), a space

$$W_h = \{\tau_h \in H(div, \Omega), \tau_h|_K \in BDM_1(K)$$
$$\forall K \in T_h\} \tag{47}$$

$$V_h = \{v_h \in L_{dis}^2(\Omega), v_h|_K \in P_0(K) \; \forall K \in T_h\}. \tag{48}$$

We chose finite dimensional subspace $H_0^h(div, \Omega) \subset H_0(\text{div}, \Omega)$.

A mixed finite element approximation of (12)-(13) is defined by

Find $\sigma_h \in W_h$ and $u_h \in V_h$ such that

$$\int_\Omega (E : \sigma_h): \tau_h dx + \int_\Omega u_h.div\tau_h dx = \int_{\Gamma_u} \bar{u} \, \tau_h.n \, d\Gamma$$
$$\forall \tau_h \in H_0^h(div, \Omega) \tag{49}$$

$$\int_\Omega v_h.div\sigma_h dx + \int_\Omega f.v_h dx = 0 \; \forall v_h \in V_h. \tag{50}$$

We obtain a system of linear equations

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}\begin{pmatrix} T \\ U \end{pmatrix} = \begin{pmatrix} S \\ L \end{pmatrix}. \tag{51}$$

Where $S = [s_m], L = [l_m]$.

The matrix associated for the system (51) is symmetric indefinite. We use the iterative methods Minimum Residual Method (MINRES) for solving the symmetric system.

IV.   NUMERICAL SIMULATIONS

Example 1. Circular Void in a Finite Plate

Here a void of radius 0.3 is placed in the center of a plate of size 3×3 which is subjected to a unit stress in the y-direction.

The stress plot for σ_{yy} s is in excellent agreement with the expected 3σ stress concentration at the edges of the hole.
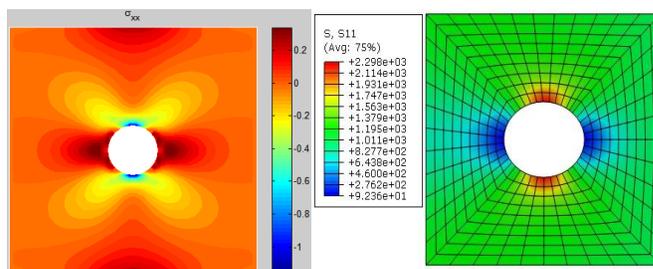

Fig. 3.   Stress Solution Σ_{xx} By Mixed Finite Element Method (Left) And Stress Solution (Right) Computed By ABAQUS.
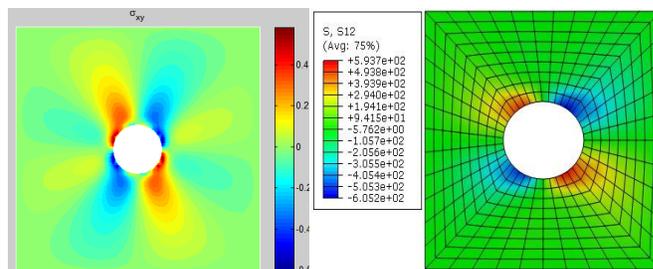

Fig. 4.   Stress Σ_{xy} By Mixed Finite Element Method (Left) And Stress Solution (Right) Computed By ABAQUS.
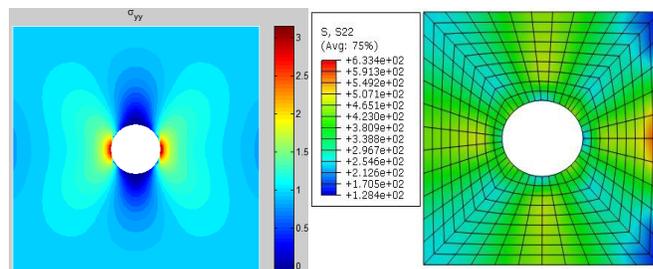

Fig. 5.   Stress Σ_{yy} By Mixed Finite Element Method (Left) And Stress Solution (Right) Computed By ABAQUS.
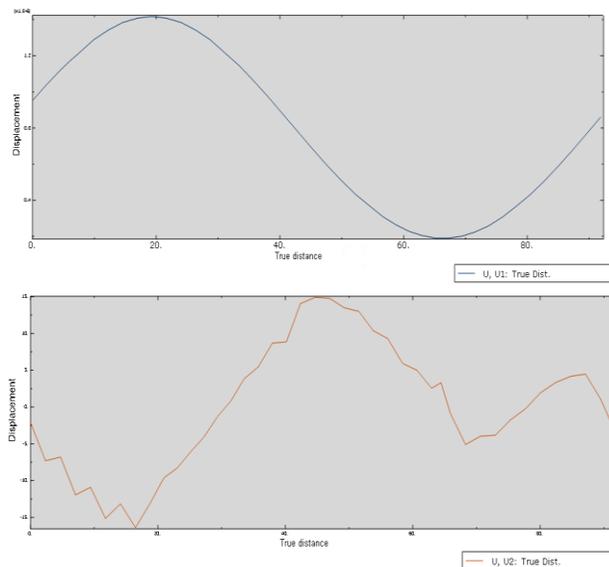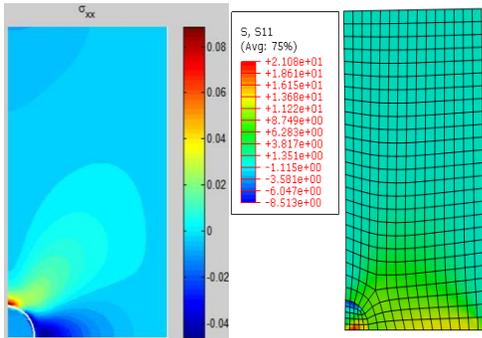

Fig. 6.   Curve Of The Displacement Ux And Uy Along Hole In A Finite Plate

Example 2. Circular Inclusion in a Finite Plate

Here an inclusion with E = 70gpa and $v = 0.3$ and radius 0.5 is placed in the center of a plate of size 6 x 10 with E = 50gpa and $v = 0.3$ which is subjected to a unit tension in the y-direction.

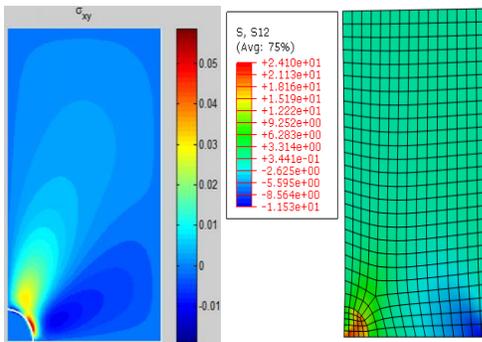Fig. 7.   Stress Σ_{xx} By Mixed Finite Element Method (Left) And Stress Solution (Right) Computed By ABAQUS.

Fig. 8.   Stress Σ_{xy} By Mixed Finite Element Method (Left) And Stress Solution (Right) Computed By ABAQUS.
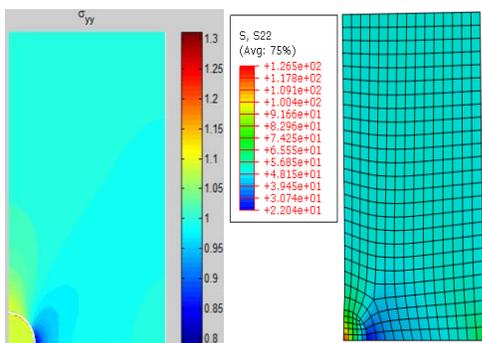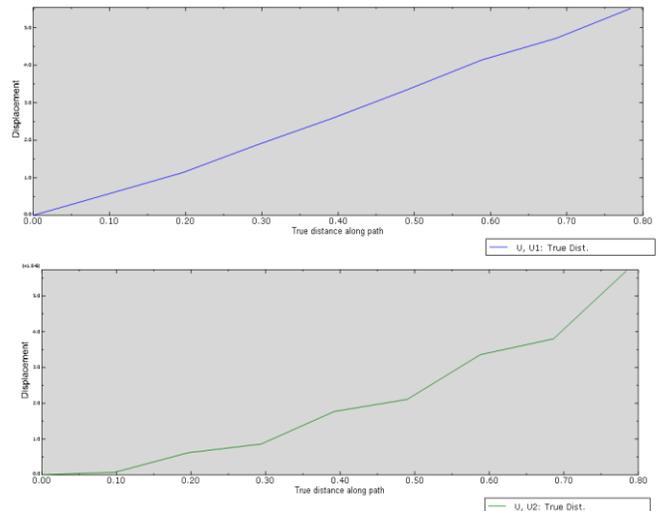

Fig. 9.   Stress Σyy By Mixed Finite Element Method (Left) And Stress Solution (Right) Computed By ABAQUS.

Fig. 10. Curve Of Displacement Ux And Curve Of Displacement Uy Along Inclusion In A Finite Plate.

V.   CONCLUSION

We were interested in this work in the numeric solution for equilibrium equations. It includes algorithms for discretization by mixed finite element methods. The discrete scheme allows the utilization of BDM_1 for the stress tensor and piecewise constant elements for the displacement. Our results agree with ABAQUS.

Numerical results are presented to see the performance of the method, and seem to be interesting by comparing them with other recent results.

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