Quartic approximation of circular arcs using equioscillating error function

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***Abstract*—A high accuracy quartic approximation for circular arc is given in this article. The approximation is constructed so that the error function is of degree 8 with the least deviation from the x-axis; the error function equioscillates 9 times; the approximation order is 8. The numerical examples demonstrate the efficiency and simplicity of the approximation method as well as satisfying the properties of the approximation method and yielding the highest possible accuracy.**

***Keywords*—Bézier curves; quartic approximation; circular arc; high accuracy; approximation order; equioscillation; CAD**

I. Introduction

The use of parametric representation of curves is convenient in the field of CAD. Especially, the parametric methods approach allows us to make use of some of the properties that are not available in the approximation of functions. For example, in [16], the idea that a parametric representation of a curve is not unique has been used to improve the order of approximation by polynomial curves of degree n from $n+1$ to $2n$. Also, the parametric form makes use of the geometric properties of the curve in design and modelling. In this paper, given the circular arc $c : t \mapsto (\cos(t), \sin(t))\,,\ -\theta \leq t \leq \theta$, where $\theta \in [-\pi, \pi]$, see Fig. 1, the geometric symmetries of the circle will be utilized to properly select the Bézier points in order to represent the quartic Bézier curve that has high order of approximation of 8 and possesses "the best" features.

A circle can be represented using rational Bézier curves and can be approximated by polynomial curves. Therefore, approximating a circular arc by polynomial curves with highest possible accuracy is a very important issue. It is needed for the construction of any CAD system. To approximate the circle c, there is a need to find a parametrically defined polynomial curve $p : t \mapsto (x(t), y(t))\,,\ 0 \leq t \leq 1$, where $x(t), y(t)$ are polynomials of degree 4, that approximates c with "minimum" error. Many researchers have tackled this issue using different norms and methods, see [2], [3], [4], [5], [6], [9], [10], [14], [16], [18]. For details and numerical comparisons with these works, see section 6. The proper function to measure the error between p and c is the Euclidean error function:

$$E(t) := \sqrt{x^2(t) + y^2(t)} - 1. \tag{1}$$

The square root limits the possibility of further progress. Thus, to avoid radicals, the squares of the components of the parametrization to the circle are used. So, the Euclidean error function $E(t)$ is replaced by the following error function

$$e(t) := x^2(t) + y^2(t) - 1. \tag{2}$$

This replacement makes sense because both $E(t)$ and $e(t)$ attain their roots and reach their extrema at the same parameters.

More precisely, **the approximation problem** in this paper is to find $p : t \mapsto (x(t), y(t))\,,\ 0 \leq t \leq 1$, where $x(t), y(t)$ are of degree 4, that approximates c and satisfies the following conditions:

1) p minimizes $\max_{t\in[0,1]}\ |e(t)|$,
2) $e(t)$ equioscillates 9 times over $[0, 1]$,

Note that condition (2) implies that p approximates c with order 8. We impose a priori these conditions, because they will be used to determine the values of the parameters that are used for geometric design of the circular arc.

The term approximation order is used in the context of Lagrange interpolation: p approximates c with order m if there exists parameters $t_1, t_2, \ldots, t_m$ in $[0, 1]$ satisfying $p(t_i) - c(t_i) = 0$, for all $i = 1, 2, \ldots, m$. This is a special case of the more general definition of order of approximation for the Hermite type including derivatives at the interpolated points.

We let the angle θ be as large as possible in order to approximate the largest circular arc with this specified error. Thereafter, the angle θ has to be scaled by a factor that also combined with a reduction in the uniform error, see the last conclusions and open problems' section.

This paper is organized as follows. Some preliminaries are given in section 2. The quartic Bézier curve of least deviation is presented and proved in section 3, and the properties are presented in section 4. All possible quartic Bézier curves of least deviation are presented in section 5. Conclusions are given in section 6.

II. Preliminaries

The notations $(x(t), y(t))$ and $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ are used to represent parametric equations, and similarly points will be used in this article.

In this paper, the curve $p(t)$ is given in Bézier form, see Fig. 2 for possible Bézier points of quartic Bézier curve. The Bézier curve $p(t)$ of degree 4 is given by, see [11]

$$p(t) = \sum_{i=0}^{4} p_i B_i^4(t) =: \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad 0 \leq t \leq 1, \tag{3}$$

where p_0, p_1, p_2, p_3 and p_4 are the Bézier points, and $B_0^4(t) = (1-t)^4,\ B_1^4(t) = 4t(1-t)^3,\ B_2^4(t) = 6t^2(1-t)^2,\ B_3^4(t) = 4t^3(1-t)$ and $B_4^4(t) = t^4$ are the Bernstein polynomial basis of degree 4.

Since it is intended to represent the arc with a polynomial curve with minimum error, it is not important if the errors occur at the endpoints or anywhere else; it is important to maintain this disruption as low as possible there where the error occurs. In some other schemes, it is necessary that the approximating Bézier curve is G^k-continuous at the end points, see [20]. To represent a circular arc, the Bézier points are selected to take advantage of the symmetry properties of the circle. As the scheme in this paper is built on the idea of minimizing the error over all of the segment $[0,1]$, therefore, the right choice for the beginning control point p_0 is as follows $p_0 = (-\alpha_0 \cos(\theta),\ -\beta_0 \sin(\theta))$, where values of α_0 and β_0 could but should not be the same. For symmetry reasons, the right choice for the end control point p_4 is as follows $p_4 = (-\alpha_0 \cos(\theta),\ \beta_0 \sin(\theta))$. Let $p_1 = (\gamma,\ -\zeta)$ then by symmetry $p_3 = (\gamma,\ \zeta)$. For symmetry reasons, the point p_2 must be on the x-axis, and thus it has the form $p_2 = (\xi,\ 0)$. Using the substitution $\alpha = \alpha_0 \cos(\theta),\ \beta = \beta_0 \sin(\theta)$, then the proper options for the Bézier points should be, see Fig. 2,

$$p_0 = \begin{pmatrix} -\alpha \\ -\beta \end{pmatrix}, \quad p_1 = \begin{pmatrix} \gamma \\ -\zeta \end{pmatrix}, \quad p_2 = \begin{pmatrix} \xi \\ 0 \end{pmatrix},$$

$$p_3 = \begin{pmatrix} \gamma \\ \zeta \end{pmatrix}, \quad p_4 = \begin{pmatrix} -\alpha \\ \beta \end{pmatrix}. \tag{4}$$

In order to have the Bézier curve p begin in the third quadrant, go counter clockwise through fourth and first quadrants and end in the second quadrant as the circular arc c, the following conditions should be fulfilled

$$\alpha, \beta, \gamma, \zeta > 0,\ \xi > 1. \tag{5}$$

The Bézier curve $p(t)$ in (3) with the Bézier points in (4) is arranged as follows

$$p(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \qquad 0 \le t \le 1. \tag{6}$$

$$= \begin{pmatrix} -\alpha\left(B_0^4(t) + B_4^4(t)\right) + \gamma(B_1^4(t) + B_3^4(t)) + \xi B_2^4(t) \\ \beta\left(B_4^4(t) - B_0^4(t)\right) + \zeta\left(B_3^4(t) - B_1^4(t)\right) \end{pmatrix}.$$

There are 5 parameters $\alpha, \beta, \gamma, \zeta, \xi$ that will be used to have the polynomial curve p comply with the conditions of the approximation problem by substituting $x(t)$ and $y(t)$ into $e(t)$ and solving the resulting equation using a computer algebra system. Thereafter, it is shown that these values satisfy the approximation conditions; this is carried out in the following section.

III. The Quartic Bézier curve of least deviation

In the following theorem, the values of $\alpha,\ \beta,\ \gamma,\ \zeta,\ \xi$ that meet the terms of the approximation problem are given.

Theorem 1: The Bézier curve (6) with the Bézier points in (4), wherein

$$\begin{aligned} \alpha = \alpha^* &:= 0.91658426813952, \\ \beta = \beta^* &:= 0.40949454135449, \\ \gamma = \gamma^* &:= 0.00389865026306327, \\ \zeta = \zeta^* &:= 2.164585487675, \\ \xi = \xi^* &:= 2.9773929563972596 \end{aligned} \tag{7}$$

fulfils the following three conditions: p minimizes the infinity norm of th error function $\max_{t\in[0,1]} |e(t)|$ and approximates c with order 8, and the error function $e(t)$ equioscillates 9 times in $[0,1]$. The error functions satisfy:

$$-\frac{1}{2^7} \le e(t) \le \frac{1}{2^7}, \quad -\frac{1}{2^7(2-\epsilon)} \le E(t) \le \frac{1}{2^7(2+\epsilon)}, \tag{8}$$

where $\epsilon = \max_{0\le t\le 1} |E(t)| \approx 2^{-8}, \forall t \in [0,1]$.

Proof: Substituting $x(t)$ and $y(t)$ from equation (6) into the error function $e(t)$ in (2) and doing thereby some simplifications yields to the following formulation

$$\begin{aligned} e(t) &= t^8\left(4\alpha^2 + 32\alpha\gamma + 64\gamma^2 - 24\alpha\xi - 96\gamma\xi + 36\xi^2\right) \\ &+ t^7\left(-16\alpha^2 - 128\alpha\gamma - 256\gamma^2 + 96\alpha\xi + 384\gamma\xi - \right. \\ &\left. 144\xi^2\right) \\ &+ t^6\left(40\alpha^2 + 16\beta^2 + 272\alpha\gamma + 448\gamma^2 - 192\alpha\xi - \right. \\ &\left. 624\gamma\xi + 216\xi^2 - 64\beta\zeta + 64\zeta^2\right) \\ &+ t^5\left(-64\alpha^2 - 48\beta^2 - 368\alpha\gamma - 448\gamma^2 + 240\alpha\xi + \right. \\ &\left. 528\gamma\xi - 144\xi^2 + 192\beta\zeta - 192\zeta^2\right) \\ &+ t^4\left(72\alpha^2 + 68\beta^2 + 320\alpha\gamma + 272\gamma^2 - 180\alpha\xi - \right. \\ &\left. 240\gamma\xi + 36\xi^2 - 240\beta\zeta + 208\zeta^2\right) \\ &+ t^3\left(-56\alpha^2 - 56\beta^2 - 176\alpha\gamma - 96\gamma^2 + 72\alpha\xi + \right. \\ &\left. 48\gamma\xi + 160\beta\zeta - 96\zeta^2\right) \\ &+ t^2\left(28\alpha^2 + 28\beta^2 + 56\alpha\gamma + 16\gamma^2 - 12\alpha\xi - 56\beta\zeta + \right. \\ &\left. 16\zeta^2\right) \\ &+ t\left(-8\alpha^2 - 8\beta^2 - 8\alpha\gamma + 8\beta\zeta\right) + (\alpha^2 + \beta^2 - 1). \end{aligned}$$

The last one is a polynomial of degree 8. The substitution of the values of $\alpha = \alpha^*,\quad \beta = \beta^*,\quad \gamma = \gamma^*,\quad \zeta = \zeta^*$ and $\xi = \xi^*$ from (7)-(9) and doing some simplifications leads to

$$e(t) = 256\ t^8 - 1024\ t^7 + 1664\ t^6 - 1408\ t^5 + 660\ t^4 - 168\ t^3 + 21\ t^2 - t + \frac{1}{128}, \quad t \in [0,1].$$

Making the substitution $t = \frac{u+1}{2}$ reduces the error function to the following form

$$e(u) = \frac{1}{128} - \frac{1}{4}\ u^2 + \frac{5}{4}\ u^4 - 2\ u^6 + u^8, \quad u \in [-1,1].$$

We know that the last polynomial is the monic Chebyshev polynomial $\tilde{T}_8(u),\ u \in [-1,1]$, which is the unique polynomial of degree 8 that equioscillates 9 times between $\pm\frac{1}{2^7}$ for all $u \in [-1,1]$ and has the least deviation from the x-axis, see [23]. This shows that p satisfies the conditions of the approximation problem. Now it is time to show the error formula for $E(t)$. The error function $e(t)$ minimized is related to the Euclidean error $E(t)$ by the following formula

$$\begin{aligned} e(t) &= x^2(t) + y^2(t) - 1 \\ &= (\sqrt{x^2(t)+y^2(t)}+1)\ (\sqrt{x^2(t)+y^2(t)}-1) \\ &= (2+E(t))\ E(t). \end{aligned}$$

Thus

$$E(t) = \frac{e(t)}{2+E(t)}.$$

Substituting the bounds of $e(t)$ gives

$$-\frac{1}{2^7(2-\epsilon)} \le E(t) \le \frac{1}{2^7(2+\epsilon)},$$
$$\text{where } \epsilon = \max_{0\le t\le 1} |E(t)| \approx 2^{-8}, \quad t \in [0,1].$$

This proves Theorem 1.

The circular arc and the approximating Bézier curve are graphed in Fig. 3. The difference between the curve and the approximation is not recognizable by the human eyes; Fig. 4 shows the corresponding error.

One would not, in general, expect a quartic polynomial to approximate almost $8/9$th the circle more accurately than this approximation. In the following section, the properties of the approximating Bézier curve are given.

IV. Properties of the quartic Bézier curve

In this section, some of the properties of the roots and the extrema of the error functions are verified. These properties characterise the approximating quartic Bézier curve. The first one is about the roots of the error functions $e(t)$ and $E(t)$ that are given in the following proposition.

Proposition I: The zeros of the error functions $e(t)$ and $E(t)$ are:

$$t_1 = \frac{1}{2}(1+\cos(\frac{\pi}{16})) = 0.9904, t_2 = \frac{1}{2}(1+\cos(\frac{3\pi}{16})) = 0.9157$$
$$t_3 = \frac{1}{2}(1+\sin(\frac{3\pi}{16})) = 0.7778, t_4 = \frac{1}{2}(1+\sin(\frac{\pi}{16})) = 0.5976,$$
$$t_5 = \frac{1}{2}(1-\sin(\frac{\pi}{16})) = 0.4025,$$
$$t_6 = \frac{1}{2}(1-\sin(\frac{3\pi}{16})) = 0.222215,$$
$$t_7 = \frac{1}{2}(1-\cos(\frac{3\pi}{16})) = 0.08427,$$
$$t_8 = \frac{1}{2}(1-\cos(\frac{\pi}{16})) = 0.009607.$$

These roots also satisfy

$$t_i + t_j \;=\; 1, \quad \text{for} \quad i+j=9.$$

Proof: The substitution of t_i in $e(t)$ gives $e(t_i) = 0,\ i = 1,\ldots,8$. These are all zeros, since $e(t)$ is a polynomial of degree 8. The error function $E(t)$ has the same roots as $e(t)$ because $E(t) = 0$ iff $\sqrt{x^2(t)+y^2(t)} = 1$ iff $x^2(t)+y^2(t) = 1$ iff $e(t) = 0$.

The approximating quartic Bézier curve p in Theorem 1 and the circular arc c intersect at the points $p(t_i) = c(t_i),\ i = 1,\ldots,8$.

In the following proposition, the extreme values are given.
Proposition II: The extreme values of e(t) and $E(t)$ occur at

$$\tilde{t}_0 = 1,\ \tilde{t}_1 = \frac{1}{2}(1+\cos(\frac{\pi}{8})) = 0.9619,$$
$$\tilde{t}_2 = \frac{1}{2}(1+\frac{1}{\sqrt{2}}) = 0.8536,\ \tilde{t}_3 = \frac{1}{2}(1+\sin(\frac{\pi}{8})) = 0.6913,$$
$$\tilde{t}_4 = \frac{1}{2},\ \tilde{t}_5 = \frac{1}{2}(1-\sin(\frac{\pi}{8})) = 0.3087,$$
$$\tilde{t}_6 = \frac{1}{2}(1-\frac{1}{\sqrt{2}}) = 0.1465,\ \tilde{t}_7 = \frac{1}{2}(1-\cos(\frac{\pi}{8})) = 0.0380602,$$
$$\tilde{t}_8 = 0.$$

These parameters satisfy the equality:

$$\tilde{t}_i + \tilde{t}_j = 1, \quad \text{for} \quad i+j = 8.$$

Proof: Differentiating $e(t)$ gives a polynomial of degree 7. Substituting $\tilde{t}_1,\ldots,\tilde{t}_7$ gives $e'(\tilde{t}_i) = 0,\ i = 1,\ldots,7$. Since $e'(t)$ is of degree 7 then these are all interior critical points. Checking at the end points adds $\tilde{t}_0 = 1,\ \tilde{t}_8 = 0$ to the critical points. Since for $t \in [0,1]$: $1-\frac{1}{128} \le x^2(t)+y^2(t) \le 1+\frac{1}{128}$, thus $\sqrt{x^2(t)+y^2(t)} \ne 0,\ \forall t \in [0,1]$. Differentiating $E(t)$ and equating to 0 gives $\frac{e'(t)}{\sqrt{x^2(t)+y^2(t)}} = 0$ iff $e'(t) = 0$. Thus $e(t)$ and $E(t)$ attain the extrema at the same values. This completes the proof of the proposition.

The difference in the values of $E(\tilde{t}_i)$ for odd and even i's is because $e(t)$ equioscillates between $\pm\frac{1}{128}$ and $\frac{1}{2^7(2-\epsilon)} \le E(t) \le \frac{1}{2^7(2+\epsilon)}$, where $\epsilon = max_{0\le t\le 1}|E(t)|$.
Proposition III: the values of $e(t)$ and $E(t)$ at $\tilde{t}_i$'s are given by:

$$e(\tilde{t}_{2i}) = \frac{1}{128}, i = 0,\ldots,4, \qquad e(\tilde{t}_{2i+1}) = \frac{-1}{128}, i = 0,\ldots,3.$$
$$E(\tilde{t}_{2i}) = 0.003899, i = 0,\ldots,4,$$
$$E(\tilde{t}_{2i+1}) = -0.003914, i = 0,\ldots,3.$$

Therefore,

$$\frac{-1}{128} \le e(t) \le \frac{1}{128} = 2(0.003906),\ t \in [0,1],$$
$$-0.00391391 \le E(t) \le 0.003899,\ t \in [0,1].$$

Proof: Direct substitution in the error functions leads to the equalities. The details of the proof of the proposition are left to the reader.
As a consequence of Theorem 1, we have the following proposition regarding the error at any $t \in [0,1]$.
Proposition IV: For every $t \in [0,1]$, the errors of approximating the circular arc using the quartic Bézier curves in Theorem 1 are given by:

$$e(t) = 256t^8 - 1024t^7 + 1664t^6 - 1408t^5 + 660t^4 - 168t^3 + 21t^2 - t + \frac{1}{128},\ \forall t \in [0,1].$$

Proof: Direct consequence of Theorem 1. The details of the proof of the proposition are left to the reader.

Using the relation between $E(t)$ and $e(t)$, we get:

$$E(t) \simeq 128t^8 - 512t^7 + 832t^6 - 704t^5 + 330t^4 - 84t^3 + \frac{21}{2}t^2 - \frac{1}{2}t + \frac{1}{256}, \quad \forall t \in [0,1].$$

To get the solution in Theorem 1, some conditions were imposed on $\alpha,\ \beta,\ \gamma,\ \zeta$ and ξ in (5). These conditions give the Bézier curve with $\alpha = \alpha^*, \quad \beta = \beta^*, \quad \gamma = \gamma^*, \quad \zeta = \zeta^*$ and $\xi = \xi^*$ that represents the circular arc from third to second quadrants passing through the fourth and first quadrants generated counter clockwise, see Fig. 3. However, if these conditions on $\alpha,\ \beta,\ \gamma,\ \zeta$ and ξ are removed, there will be other possible solutions. These are given in the following section.

V. All possible quartic Bézier curves

The following theorem lists all the possible Bézier curves.

Theorem 2: By removing the conditions on $\alpha, \beta, \gamma, \zeta$ and ξ in (5) and reinvestigating the approximation problem, then the problem has 32 solutions; 24 of these solutions are complex and the other 8 solutions are real; 4 real solutions are not admissible because they have the opposite direction for the tangent; the other 4 real solutions are geometrically feasible and satisfy the conditions of the approximation problem. These solutions are sign multiple of the solution in Theorem 1 and are summarized in table 1.

Proof: The first solution has been confirmed in Theorem 1. To confirm the other 3 cases, we consider the error function $e(t)$ and do some simplifications and substitutions as in Theorem 1 to get the monic Chebyshev polynomial of degree 8. This polynomial possesses the properties of the approximation problem. The details of the proof are left to the reader.

Remarks:

1) All of the solutions in Table 1 are related to each other. The second solution coincides with the first solution, but generated clockwise. The third and fourth solutions are reflections of the first solution around the y-axis, generated counter clockwise and clockwise, respectively.
2) The sign of α is the same as the signs of γ and ξ. If the sign of α is positive then the curve begins (ends) in the second quadrant through the first and fourth quadrants and ends (begins) in the third quadrant, and if it is negative then the curve begins (ends) in the first quadrant through the second and third quadrants and ends (begins) in the fourth quadrant.
3) The sign of β is the same as the sign of ζ.

The roots and extreme values of $e(t)$ and $E(t)$ for all the solutions in Table 1 are given in the following proposition.

Proposition V: The solutions in Table 1 have the following properties:

1) The roots of the error functions $e(t)$ and $E(t)$ for all of the solutions in Table 1 are the same as in Proposition I.
2) The extreme values of $e(t)$ and $E(t)$ for all of the solutions in Table 1 occur at the same parameters that are given in Proposition II.
3) The extreme values of $e(t)$ and $E(t)$ for all of the solutions in Table 1 have the same values that are given in Proposition III.
4) The error functions $e(t)$ and $E(t),\ t \in [0,1]$ for all of the solutions in Table 1 are given by the formulas in Proposition IV.

Proof: The proofs are similar to the proofs of the similar previous cases and are left to the reader.

VI. Conclusions and Open Problems

In this article, the best uniform approximation of circular arcs with parametrically defined polynomial curves of degree 4 are explicitly given. The error function equioscillates 9 times; the approximation order is 8. Numerical examples are given to demonstrate the efficiency and simplicity of the approximation method.

The method in this paper is C^0-continuous by construction. There are methods in the literature that are G^1- and G^2-continuous, see for example [6], [9], [10], [13], [14], [16], [17], [18], [19], [22].

As future works, it is interesting to:

1) study quartic approximation with G^k-continuity, $k = 1, 2$, using equioscillating error functions and constrained Chebyshev polynomials.
2) find a way to write the Bézier points in terms of the angle θ. It would be very important to have the best approximation available for all θ perhaps by employing a semi-numerical method.
3) Apply these results in this paper to perform degree reduction of Bézier curves to get the best approximation with the minimum uniform error.
4) It would be interesting to compare our curve with the quartic exponential Euler spline defined by Schoenberg and studied by de Boor, see [7], [8], [24], [25].

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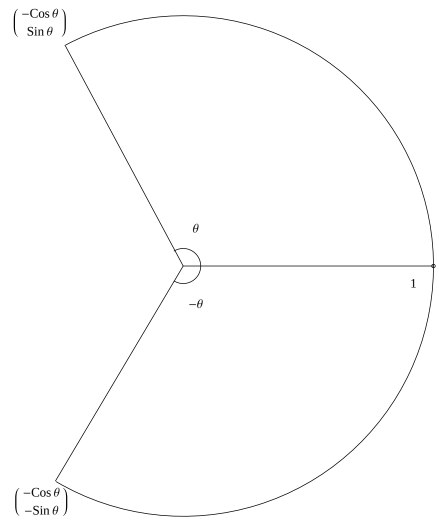

Fig. 1: A circular arc

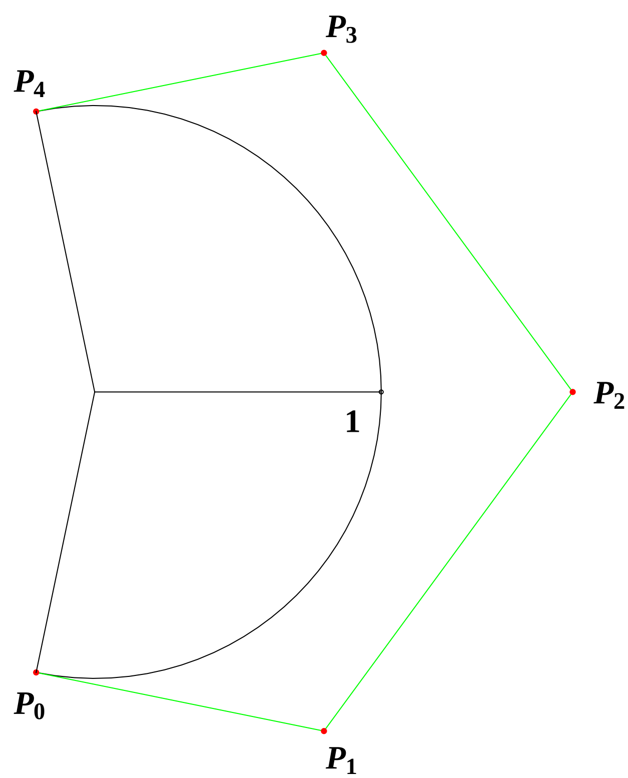

Fig. 2: Possible Bézier points of circular arc.

Solution	Sign α	Sign β	Sign γ	Sign ζ	Sign ξ	from to quadrants	generated
1st	$+$	$+$	$+$	$+$	$+$	3rd to 2nd	counter clockwise
2nd	$+$	$-$	$+$	$-$	$+$	2nd to 3rd	clockwise
3rd	$-$	$-$	$-$	$-$	$-$	1st to 4th	counter clockwise
4th	$-$	$+$	$-$	$+$	$-$	4th to 1st	clockwise

Table 1: All geometrically feasible real solutions to the approximation problem.

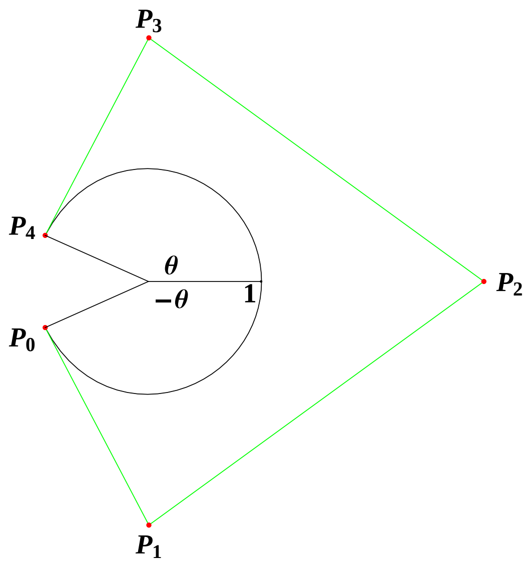

Fig. 3: Circular arc and it's quartic Bézier curve in Theorem 1.

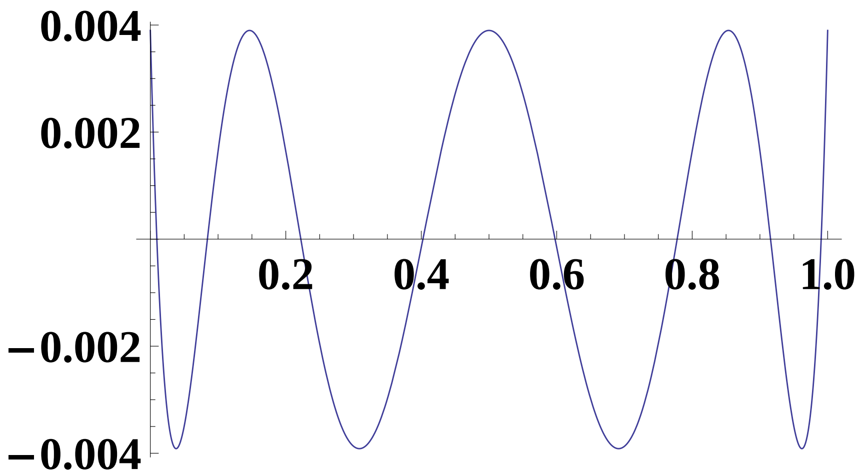

Fig. 4: Euclidean Error of the quartic Bézier curve in Theorem 1.