Computational Model for the Generalised Dispersion of Synovial Fluid

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***Abstract*—The metabolic function of synovial fluid is important to understand normal and abnormal synovial joint motion, especially if one seeks some leading causes of the degenerative joint disease. The concentration of hyaluronic acid molecules and other high molecular weight substances in the synovial fluid may be responsible to disperse the nutrients into the cartilage. The theoretical study of the convective diffusion mechanism occurring in the knee joint is presented. A flow model has been analyzed for better understanding of the convective diffusion of the viscous flow in between the articular surfaces. The governing system of partial differential equations has been solved for the Newtonian fluid with suitable matching and conditions. The analytical solution for the unsteady dispersion problem has been obtained for the better understand the phenomena of nutritional transport to synovial joint. The contributions of (convection + diffusion) on the dispersion of nutrients are investigated in detail. The dispersion coefficient has been computed for different values of the viscosity parameter. The results show that the average concentration has a negative correlation with the axial distance and the time.**

***Keywords*—Synovial Fluid; Articular Cartilage; Unsteady diffusion coefficient; Computational model**

I. INTRODUCTION

The synovial joints play a very important role in humans and animal locomotion. These joints sustain the very high loads and low friction under normal physiological condition [1]. Articular cartilage can be considered as a porous gel of proteoglycan aggregates embedded in a water swollen network of collagen fibrils. When cartilage is compressed, its interstitial fluid is forced to flow relative to the solid organic matrix and to be exuded from this matrix [2, 3]. While immature articular cartilage contains vessels that transport the nutrients. The extra-cellular fluid through diffusion and convection [4] transports nutrients from the synovial fluid [5]. The process of dispersion plays a very important role in chemical as well as in biological systems [6-11].

The generalized dispersion model is very useful and valid for all time, for examples of biomedical engineering, namely coronary artery diseases [CAD] and synovial joints [12]. In CAD The suspended particles may execute microcirculation in dispersing through the endothelium. Similarly, with synovial joints, hyaluronic acid [HA], glycoprotein, and other macromolecular components disperse from synovial fluid to cartilage. The diffusion coefficient of hyaluronan in healthy synovial fluid was on average 30% slower than expected by sample viscosity [13]. HA and other components may also execute microcirculation while dispersing from synovial fluid to cartilage. The endothelium in arteries and cartilage in synovial joints are layers of porous cells that may or may not deform.

Synovial fluid plays the key role in the lubrication of the joints, and also for the provision of nutrients and removal of metabolites from the avascular articular cartilage. Boosted lubrication, the process of imbibition and exudation increase the concentration of the hyaluronic acid molecules in synovial fluid [14]. The increased concentration of hyaluronic acid will give rise to the increase in the viscosity of the synovial fluid according to the Wegamirs findings. The macroscopic behavior of the particulate suspension can thus approximate to the homogeneous fluid of greater viscosity than the suspending medium [15]. The artificial joints functioning depend on the dispersion of hyaluronic acid and nutritional transport from synovial fluid to the joints [23-25].

Various attempts have been made by the researchers to investigate the characteristics of dispersion in fluid dynamical situations. Taylor studied under effect the real-time action of molecular diffusion and variation of the velocity of solvent on the dispersion of solute [18]. Gill et al [26] has been obtained the exact solution for the dispersion coefficients. Aris et al [19] study the dispersion process under restricted parameter describing the concentration of solute in terms of its moments in the direction of flow.

In this paper, an effort has been constituted to examine the generalized dispersion of hyaluronic acid particles and other proteins of synovial fluid for the endurance of the cartilage. The joint replacement leads into the rheologically modified lubricant, called periprosthetic fluid, and behaves almost as water low viscosity, Newtonian fluid [20]. It advises us to represent synovial fluid by the viscous fluid in between two approaching rigid plane surfaces. Thus, to know whether an artificial joint works efficiently or not it is essential to look into the dispersion phenomena using Newtonian lubricant. At the porous boundary, Beavers and Joseph boundary conditions with a slightly modified form have been used. For the viscosity of the intermission, the dispersion coefficient is found from the diffusion equation using the generalized hypothesis. The dominant dispersion coefficients for mean concentration have been analyzed in details. It has been set up that the viscosity coefficient decreases as the diffusion coefficient increases. The effects have also been obtained for mean concentration distribution of several values of viscosity. It is observed that viscosity decreases with the mean concentration distribution and increases with the diffusion coefficient.

II. Mathematical Formulation and Solution

The knee joint plays a very important role in human locomotion. It plays an essential for the movement of body as well as carries the many times of the body weight in the horizontally and vertically direction during running and walking almost frictionless. The knee joint is one of the largest and most complex joints in the body. The knee joins the thighbone (femur) to the shinbone (tibia). Tendons connect the knee bones to the leg muscles that move the knee joint. Ligaments join the knee bones and provide stability to the knee in preventive and self-corrective ways. The anterior cruciate ligament prevents the femur from sliding backward on the tibia (or the tibia sliding forward on the femur).The posterior cruciate ligament prevents the femur from sliding forward on the tibia (or the tibia from sliding backward on the femur). The medial and lateral collateral ligaments prevent the femur from sliding side to side.

The configuration of bearing model for analysis consists of two rectangular plates of infinite length (not shown in the figure) in x direction. The surfaces are kept apart by a fluid film of thickness 2h. Introducing the usual assumption of lubrication theory in the Navier–Stokes equation of motion and neglecting the variation of pressure normal to very thin film of lubrication, the following differential equations are obtained for pressure (p) in the fluid film region.

$$-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2}=0 \qquad (1)$$

$$0=-\frac{\partial p}{\partial y}$$

Boundary Conditions:

$$u=-\frac{\sqrt{\emptyset}}{\alpha}\frac{\partial u}{\partial y}\ \ at\ y=h$$

$$u=0 \quad at\ y=0 \qquad (2)$$

$$p=0\ at\ x=\pm\frac{x_s}{2}$$

where, $\emptyset$, $\alpha\ and\ \mu$ are porosity, slip parameter and viscosity coefficient respectively

$$\text{Using } \eta=y/h,\ \frac{dp}{dx}=\frac{\mu}{h}\frac{d^2u}{d\eta^2} \qquad (3)$$

$$u=\frac{dp}{dx}\left[\frac{h\eta^2}{2\mu}-\eta\left(\frac{h}{2\mu}+\frac{\sqrt{\emptyset}}{\alpha}\frac{h}{\mu}\right)\Big/\left(1+\frac{\sqrt{\emptyset}}{\alpha h}\right)\right] \qquad (4)$$

$$u^*=\frac{u-\bar{u}}{\bar{u}}=3\eta^2-6\eta\beta-1 \qquad (5)$$

where, $\beta=\left(\frac{1}{2}+\sigma\right)\left(\frac{\alpha}{\alpha+\sigma}\right), \sigma=\frac{\sqrt{\emptyset}}{\alpha}$, u^* is the average velocity in fluid film region.

A simple mass balance between changes in concentration $c(t,x,y)$ of solutes like HA molecules in synovial fluid by convection and diffusion leads to in terms of the dispersion coefficient of making molecules in synovial fluid, got here as approximately spatially uniform.

$$\frac{\partial c}{\partial t}+(u-\bar{u})\frac{\partial c}{\partial x}=D\left(\frac{\partial^2 c}{\partial x^2}+\frac{\partial^2 c}{\partial y^2}\right) \qquad (6)$$

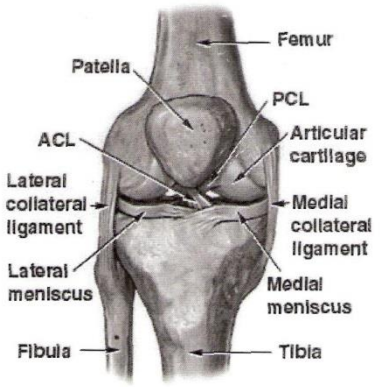

Fig. 1. Right Knee – Anterior view with Patella Tendon Removed

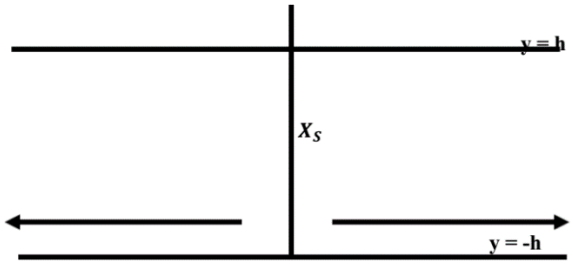

Fig. 2. Parallel Plate Geometry for Dispersion in a synovial fluid

Where $c\ (t,x,y)$ is the concentration of the initial input of length x_s

It has been assumed that the solution concentration is c_0 at the time when the process of imbibition and exudation starts and the superficial and deep layer of articular cartilage is solute free.

$$c(0,x,y)=\ c_0\ \text{ for } |x|\le\frac{1}{2}x_s$$

$$c(0,x,y)=\ 0 \quad \text{for } |x|\ge\frac{1}{2}x_s \qquad (7)$$

$$c(t,\infty,y)=0,\ \ \frac{\partial c}{\partial y}(\tau,x,y)=0\ at\ \ y=\pm h$$

and putting in the accompanying non-dimensional system

$$\Theta=\frac{c}{c_0}, \xi=\frac{Dx}{h^2\bar{u}}, X_s=\frac{Dx_s}{h^2\bar{u}}, \eta=\frac{y}{h}, \tau=\frac{Dt}{h^2} \qquad (8)$$

The Eqn. (6) and (8) may be written in non-dimensional form as:

$$\frac{\partial\Theta}{\partial\tau}+u^*\frac{\partial\Theta}{\partial\xi}=\frac{1}{{P_e}^2}\frac{\partial^2\Theta}{\partial\xi^2}+\frac{\partial^2\Theta}{\partial\eta^2} \qquad (9)$$

$\bar{u}$ is the cross-sectional average velocity and can be defined by $\bar{u}=\frac{1}{2h}\int_{-h}^{h}u(y)dy$

where, $Pe=\bar{u}h/D$, and $\xi=X-\bar{u}\tau$ are Peclet number and the non-dimension coordinates and parameters moving with the mean velocity $\bar{u}$. The boundary and initial condition (7) takes the form

$$\Theta(0,X,\eta) = 1 \text{ for } |X| \le \tfrac{1}{2}X_s$$
$$\Theta(0,X,\eta) = 0 \text{ for } |X| \ge \tfrac{1}{2}X_s \quad (10)$$
$$\Theta(\tau,X,\eta) = 0, \ \frac{\partial\Theta}{\partial\eta}(\tau,X,\eta) = 0 \ at \ \ \eta = \pm 1$$

The solution of Eqn. (9) subject to Eqn. (10) is written as a series expansion in $\frac{\partial^k\Theta}{\partial\xi^k}$, in the form

$$\Theta = \Theta_m(\tau,\xi) + \sum_{k=1}^{\infty} f_k(\tau,\eta)\frac{\partial^k\Theta_m}{\partial\xi^k} \quad (11)$$

where, $\Theta_m(\tau,\xi) = \frac{1}{2}\int_{-1}^{1}\Theta d\eta$ (12)

Substituting (11) in Eqn. (9), we have

$$\frac{\partial\Theta_m}{\partial\tau} + U^*\frac{\partial\Theta_m}{\partial\xi} - \frac{1}{{P_e}^2}\frac{\partial^2\Theta_m}{\partial\xi^2} + \sum_{k=1}^{\infty}\left[\left(\frac{\partial f_k}{\partial\tau}\right) - \left(\frac{\partial^2 f_k}{\partial\eta^2}\right)\right]\frac{\partial^k\Theta_m}{\partial\xi^k} + U^*\frac{\partial^{k+1}\Theta_m}{\partial\xi^{k+1}} - f_k Pe^{-2}\frac{\partial^{k+2}\Theta_m}{\partial\xi^{k+2}} + f_k\frac{\partial^{k+1}\Theta_m}{\partial\tau\partial\xi^k} = 0 \quad (13)$$

where,

$$\frac{\partial\Theta_m}{\partial\tau} = \sum_{k=1}^{\infty} k_k(\tau)\frac{\partial^k\Theta_m}{\partial\xi^k}$$

where the dispersion coefficients $k_k(\tau)$ are time dependent. This form, unlike in the model of Taylor (1953) and Aris (1956) is established on the premise that the cognitive operation of distributing (is diffusive in nature right from time zero. Now Eqn. (13) is solved subject to boundary conditions:

$$\Theta_m(0,\xi) = 1 \text{ for } |\xi| \le \tfrac{1}{2}X_s \quad (14)$$
$$\Theta_m(0,\xi) = 0 \text{ for } |\xi| \ge \tfrac{1}{2}X_s \quad (15)$$
$$\Theta_m(\tau,\xi) = 0 \quad (16)$$

Using Eqn. (14) into (13), and rearranging terms, we get

$$\left[\frac{\partial f_1}{\partial\tau} - \frac{\partial^2 f_1}{\partial\eta^2} + U^* + k_1(\tau)\right]\frac{\partial\Theta_m}{\partial\xi} + \left[\frac{\partial f_2}{\partial\tau} - \frac{\partial^2 f_2}{\partial\eta^2} + U^* f_1 + f_1 k_1(\tau) + k_2(\tau) - Pe^{-2}\right]\frac{\partial^2\Theta_m}{\partial\xi^2} + \sum_{k=1}^{\infty}\left[\frac{\partial f_{k+2}}{\partial\tau} - \frac{\partial^2 f_{k+2}}{\partial\eta^2} + U^* + f_{k+1}k_1(\tau) + (k_2(\tau) - Pe^{-2}) + f_k\sum_{i=3}^{k+2} k_i(\tau) f_{k+2-i}\right]\frac{\partial^{k+2}\Theta_m}{\partial\xi^{k+2}} = 0 \quad (17)$$

We get,

$$f_1 = \frac{\eta^4}{4} - \eta^3\beta + \frac{\eta^2}{2} - \frac{13}{60} + \sum_{n=1}^{\infty} A_n e^{-{\lambda_n}^2\tau}\cos(\lambda_n\eta) \quad (18)$$

where,

$\lambda_n = n\pi$, and

$$A_n = -{\lambda_n}^{-2}\left[(-1)^n(3\beta - 4) - 6\beta{\lambda_n}^{-2}\right] \quad (19)$$

On Solving the above equations dispersion coefficient, we get

$$k_2(\tau) = Pe^{-2} - \frac{12\beta^2}{5} - \frac{86}{105} + 12\sum_{n=1}^{\infty}\frac{A_n e^{-{\lambda_n}^2\tau}}{{\lambda_n}^2} \quad (20)$$

Similarly $K_3(\tau)$, $K_4(\tau)$ and so on are obtained. The expressions are omitted here because of their lengthy expression. We find that $k_i(\tau), (i > 2)$ are negligibly small compared to $k_2(\tau)$, the generalized dispersion model reduces to

$$\frac{\partial\Theta_m}{\partial\tau} = K_2(\tau)\frac{\partial^2\Theta_m}{\partial\xi^2}$$

and the solution of above Eqn. is

$$\Theta_m = \frac{1}{2}\left\{erf\left(\frac{\frac{1}{2}X_s - \xi}{2\sqrt{T}}\right) + erf\left(\frac{\frac{1}{2}X_s + \xi}{2\sqrt{T}}\right)\right\}$$

where, $T = \int_0^{\tau} K_2(z)\,dz$

$$T = \left(Pe^{-2} - \frac{12\beta^2}{5} - \frac{86}{105}\right)\tau + 12\sum_{n=1}^{\infty}\frac{A_n(1 - e^{-{\lambda_n}^2\tau})}{{\lambda_n}^4}$$

III. Results and Discussions

The problem of generalized dispersion has been analyzed for a simplified computational model of human knee joints with representation of synovial fluid by the viscous fluid. Fig. 3 is plotted between Taylor's dispersion coefficients $k_2(\tau)$ and instantaneous time τ for different values of viscosity μ of the synovial fluid. It has been depicted from the figure that at the increase value of time the second dispersion coefficient $k_2(\tau)$ also increases. It has been depicted from the fig 3. that the dispersion coefficient increasing with decreasing values of the viscosity μ decreases. The similar result has been obtained by [16] in a dispersion of solutes in a in the laminar flow between two parallel plates by taking into consideration the homogeneous and heterogeneous reaction of the solvent with the solute. In the case of a diabetic patient as compared to normal subject the viscosity of the plasma is higher;

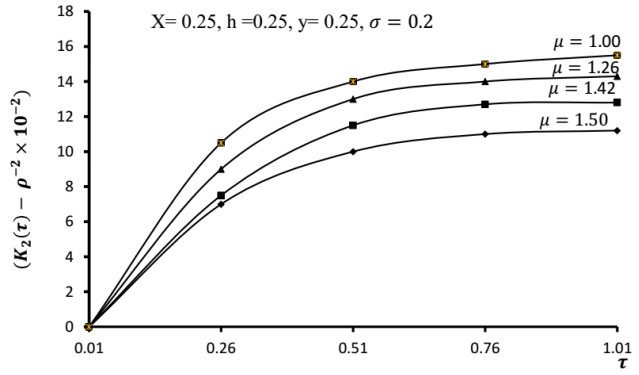

Fig. 3. Variation of dispersion coefficient with time τ for different values of the of viscosity

in turn the diffusion coefficient for the diabetic person is generally higher. Fig. 4 shows the variation of mean concentration distribution Θ_m with axial distance for different values of viscosity μ.

The Fig. 4 shows clearly that mean concentration Θ_m decreases asymptotically as axial distance approaches to infinity. It should be noted that as viscosity μ increases then mean concentration also increases [17, 21]. The concentration of hyaluronic acid molecules increase due to increase value of the viscosity of the synovial fluid, eventually it increases the apparent viscosity of the lubricant i.e. synovial fluid. Fig. 5 depict the variation between the non-dimensional concentration distribution of solute with non-dimensional time for various values of viscosity. It is clear from the figure that the mean

concentration distribution Θ_m decreases with increases values of the time τ [21, 22].

In synovial cavity, enhanced diffusion may occur in the initial phases of the movement as the solute only moves through the larger interstitial spaces. As diffusion progresses the solute may move into the smaller interstitial volumes.

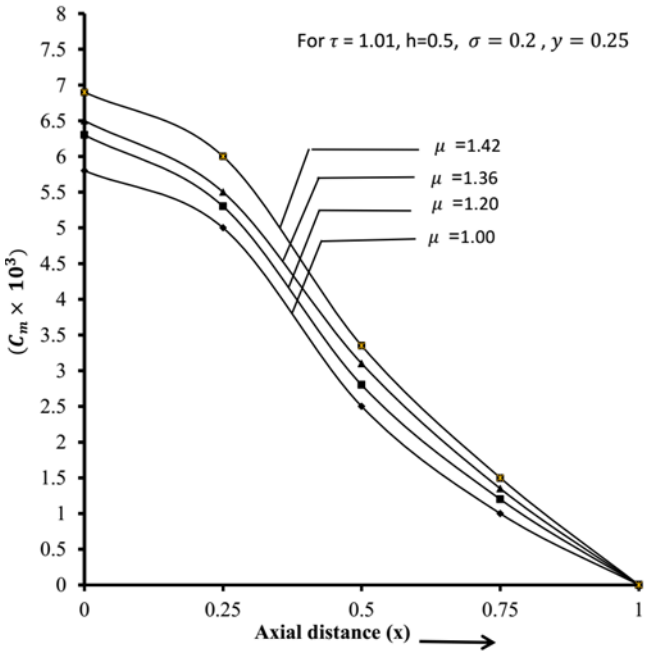

Fig. 4. Variation of mean concentration distribution with axial distance for different values of viscosity (μ)

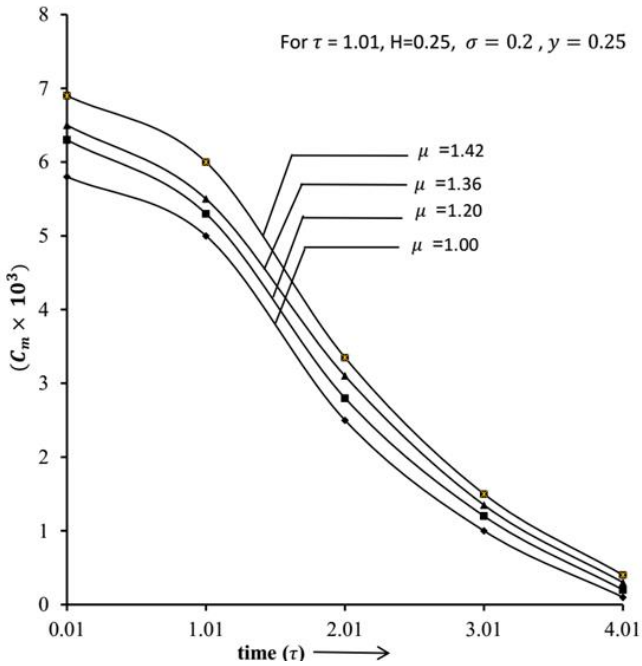

Fig. 5. Variation of mean concentration with time (τ) for different values of viscosity (μ)

IV. Conclusions

The dispersion of proteins and other nutrients from the synovial fluid to articular cartilage is studied using the [18] exact analysis of unsteady convective diffusion. It has also been observed that dispersion coefficient $k_2(\tau)$ increases with a decrease in the viscosity. It is seen that the mean concentration distribution decreases Θ_m with an increase in the axial distance and in the time and increases with increase in viscosity.

It is seen that the mean concentration distribution decreases with an increase in the time and axial distance the cells of middle area get more nutritional as compared to the peripheral area. It helps to orthopedic surgeons to check by the formula of dispersion mechanism, whether the joints functioning effectively or not. In the future, the model for unsteady convective diffusion can be used for the development of a mathematical model for the articular cartilage regeneration because the key mechanism involved in the cartilage regeneration modeling cell migration, nutrient diffusion and depletion extracellular matrix synthesis and degradation at the defect site, both spatially and temporally.

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