Nonlinear Identification and Control of Coupled Mass-Spring-Damper System using Polynomial Structures

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***Abstract*—The paper aims to identify and control the coupled mass-spring-damper system. A nonlinear discrete polynomial structure is elaborated. Its parameters are estimated using Recursive Least Squares (RLS) algorithm. Moreover, a feedback stabilizing control law based on Kronecker power is designed. Finally, simulations are presented to illustrate the effectiveness of the proposed structure.**

***Keywords*—Identification; RLS algorithm; Polynomial structure; Stabilizing control; LQR**

I. Introduction

System identification is an important tool which can be used to improve control performance [1] [2]. It is the process of developing a mathematical representation of a physical system based on observed data with sufficient accuracy.

Identification of complex systems has stilled a major problem in automatic control because there is no general method for studying high order processes. Indeed, it has received considerable attention and several types of models have been proposed during the last decades [3] [4] [5] [6] [7]. Such as Volterra model [8] [9], Wiener model [10], Hammerstein model [11], Nonlinear AutoregRessive with eXogenous input (NARX) model [12], Nonlinear AutoregRessive Moving Average with eXogenous input (NARMAX) model [13] [14], *etc.* However the elaboration of a suitable feedback stabilizing control using the proposed models remain difficult.

Nonlinear discrete polynomial structure is general enough to describe many physical systems [15] [16]. It presents the advantage to permit the use of the kronecker product and power of matrices and vectors, which allows important algebraic manipulations [17]. Moreover, it allowed to design a feedback stabilizing control law [18].

In this work, a suitable nonlinear discrete polynomial structure was elaborated. Recursive Least Square (RLS) algorithm is used for parameters estimation. The polynomial model allowed to design an efficient feedback stabilizing control law. A CMSD system illustrated the proposed nonlinear parametric estimation and structures.

This paper is organized as First, the nonlinear identification procedure is defined. Second, the feedback stabilizing control is presented. Third, the proposed identification method is applied to CMSD system and finally a conclusion is made.

II. System Identification

In automatic control applications, a compact and accurate description of the dynamic behavior of the system under consideration is needed. Nonlinear models can be constructed from theoretical modeling on the basis of *a priori* knowledge on the nature of the systems. However, these white-box models are very complex and difficult to derive because they require detailed specialist knowledge which is practically or totally unavailable in practical situation [19].

An alternative way of building models is by system identification. It is the process of improving a mathematical representation of a physical systems based on observed input/output data with sufficient accuracy which can be used to improve control performance and achieve robust fault tolerant behavior.

The identification procedure is summarized as follows:

- collection of the inputs and outputs measurements,
- selection of the model,
- choice of the identification algorithm in order to estimate the parameters that describe the model,
- validity of the obtained model is evaluated.

There are several types of models that describe complex systems. Nonlinear discrete polynomial structures is one of the most performers models. Hence, it can approach with satisfactory accuracy any analytical nonlinear system and thus ensure the mathematical description of a wide range of physical process [18] [20] [15]. Moreover, the description of polynomial systems can be simplified using the Kronecker product and power vectors and matrices.

A. Nonlinear discrete polynomial structures

We consider in this paper the discrete nonlinear polynomial systems described by a state equation of the following form [16]:

$$X_{k+1} = F(X_k) + G(X_k)\ U_k \tag{1}$$

where $F(X_k)$ and $G(X_k)$ are a polynomials vectors functions. They are given by [15]:

$$F(X_k) = \sum_{i\geq 1} A_i X_k^{[i]} \quad (2)$$

$$G(X_k) = \sum_{i\geq 0} B_i \left(I_m \otimes X_k^{[i]}\right) \quad (3)$$

with $X_k = (x_{1,k}, x_{2,k}, \ldots, x_{n,k})^T \in R^n$, $X_k^{[i]}$ is the Kronecker power of the vector X_k defined as:

$$\begin{cases} X_k^{[0]} = 1 \\ X_k^{[i]} = X_k^{[i-1]} \otimes X_k = X_k \otimes X_k^{[i-1]} \\ \text{for } i \geq 1 \end{cases} \quad (4)$$

where $\otimes$ designates the symbol of the Kronecker product, A_i and B_i are respectively $(n \times n^i)$ and $(n \times mn^i)$ matrices. I_m is the identity matrix of order m. We assume that the pair (A_1, B_0) is completely controllable.

Parametric estimation using recursive algorithms is one of the most important areas in system and signal processing. The RLS algorithm is one of the most popular ones and widely used for the parameter estimation because of his capability to approximate a large class of systems and his simplicity of implementation [21].

B. RLS algorithm

RLS algorithm allows to estimate the model parameters by minimizing a measure of the model prediction error given by [22]:

$$\varepsilon_k = y_k - \hat{y}_k \quad (5)$$

where $\hat{y}_k$ is the prediction of the scalar measured output y_k. It is given by:

$$\hat{y}_k = \hat{\theta}_k^T \psi_k \quad (6)$$

$\hat{\theta}_k$ is the vector of estimated parameters and ψ_k is the regression vector containing old inputs and outputs of the system to be identified.

The RLS algorithm can be written in following form:

$$\begin{cases} \hat{\theta}_k = \hat{\theta}_{k-1} + P_k \psi_k \varepsilon_k \\ P_k = P_{k-1} - \frac{P_{k-1}\psi_k \psi_k^T P_{k-1}}{1 + \psi_k^T P_k \psi_k} \\ \varepsilon_k = y_k - \hat{y}_k \end{cases} \quad (7)$$

with P_k is the gain matrix. It is given by:

$$P_k = \left(\sum_{i=n+1}^{k} \psi_i \, \psi_i^T\right)^{-1} \quad (8)$$

C. Performance indicators

The performance of the models is assessed using the Mean Square Error (MSE) and the Variance-Accounted-For (VAF) indicators [12]:

$$MSE = \frac{1}{N}\sum_{k=1}^{N} \left(y_{s,k} - y_k\right)^2 \quad (9)$$

$$VAF = \max\left\{1 - \frac{\operatorname{var}(y_{s,k} - y_k)}{\operatorname{var}(y_{s,k})}, 0\right\} \times 100 \quad (10)$$

where $y_{s,k}$ and y_k are respectively the system and the model output, N present the number of iterations and $\operatorname{var}(.)$ denotes the variance of a signal.

III. Nonlinear Feedback Stabilizing Control

In this section, we propose to determine a stabilizing control law of the system in the following form [18]:

$$U_k = H(X_k) \quad (11)$$

where $H(X_k)$ is an analytical vectorial function from R^n into R^m.

It is expressed by generalized Taylor series:

$$H(X_k) = -\sum_{j\geq 1} K_j X_k^{[i]} \quad (12)$$

where K_j, $j = 1, \ldots, r$ are $(m \times n^i)$ matrices. Thus, the controlled system equation can be written as [18]:

$$\begin{aligned} X_{k+1} = &\sum_{i\geq 1} A_i X_k^{[i]} \\ &- \sum_{i\geq 0}\sum_{j\geq 0} B_i \left(I_m \otimes X_k^{[i]}\right) K_j X_k^{[j]} \end{aligned} \quad (13)$$

Our objective is to determine the control function so that the stability of the null equilibrium $(X_k = 0)$ of the system. The best solution of such a problem consists in the determination of the matrices K_j, $j \in N$. The matrix K_1 is obtained using the Discrete Linear Quadratic Regulator (DLQR) state feedback design.

DLQR is one of the optimal control techniques. It takes into account the states of the dynamical system and control input to make the optimal control decisions. This is simple as well as robust [23] [24]. The discrete state equation is given by:

$$X_{k+1} = A_1 X_k + B_0 U_k \quad (14)$$

then, the state feedback control U_k is defined as:

$$U_k = -K_1 X_k \quad (15)$$

which leads to:

$$X_{k+1} = (A_1 - B_0 K_1) X_k \quad (16)$$

K_1 is derived from minimization of the cost function:

$$J(X_k) = \frac{1}{2}\sum_{i=k}^{\infty} \left(X_i^T Q X_i + U_i^T R U_i\right) \quad (17)$$

where Q and R are positive semi-definite and positive definite symmetric constant matrices, respectively. The DLQR gain vector K_1 is given by:

$$K_1 = \left(R + B_0^T P B_0\right)^{-1} B_0^T P A_1 \tag{18}$$

where P is a positive definite symmetric constant matrix obtained from the solution of matrix Algebraic Riccati Equation (ARE):

$$\begin{array}{c} A_1^T P A_1 - P + Q \\ -A_1^T P B_0 \left(R + B_0^T P B_0\right)^{-1} B_0^T P A_1 = 0 \end{array} \tag{19}$$

However, the matrices K_j, for $j \geq 2$, are given by the following relation [18]:

$$K_j = -B_0^+ \left(A_j + \sum_{i=1}^{j-1} B_i \left(K_{1-i} \otimes I_{n^i}\right)\right) \tag{20}$$

where B_0^+ designates the Moore-Penrose pseudo-inverse of the matrix B_0.

IV. Illustrative Example: Coupled Mass-Spring-Damper System

A. CMSD system description

The CMSD system, shown in Figure 1, is composed of two nonlinear springs, two weights and two dampers. Since the upper mass m_1 is attached to both springs, there are two nonlinear springs restoring forces acting upon it: an upward force f_{r1} exerted by the elongation, or compression, x_1 of the first spring; an upward force f_{r2} from the second spring resistance to being elongated, or compressed, by the amount $(x_2 - x_1)$.

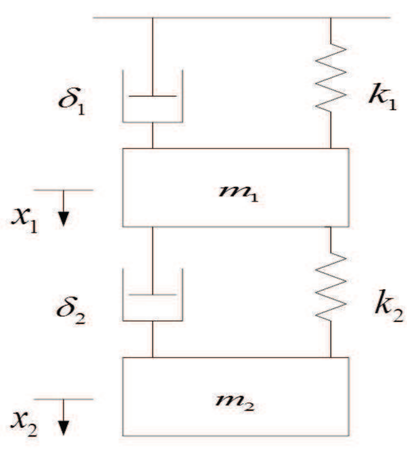

Fig. 1 – Mechanical model of the CMSD system

The second mass m_2 only feels the nonlinear restoring force from the elongation, or compression, of the second spring. Allowing the system to come and to rest in equilibrium, we measure the displacement of the center of mass of each weight from equilibrium, as a function of time, and denote these measurement by x_1 and x_2 respectively. System parameters are presented in Table 1 [25].

TABLE I – Parameter Description of CMSD System

Parameter	Description	Value
$k(N/m)$	spring constant	$k_1 = \frac{2}{5}$, $k_2 = 1$
$x(m)$	displacement	x_1, x_2
$m(Kg)$	mass of the weight	$m_1 = 1$, $m_2 = 2$
$\delta(Ns/m)$	damping coefficient	$\delta_1 = \frac{1}{10}$, $\delta_2 = \frac{1}{5}$
μ	nonlinear coefficient	$\mu_1 = \frac{1}{6}$, $\mu_2 = \frac{1}{10}$

1) Mathematical model: The continuous nonlinear equations of the CMSD system are given by:

$$\begin{cases} m_1 \ddot{x}_1 = -\delta_1 \dot{x}_1 - k_1 x_1 + \mu_1 x_1^3 - k_2 (x_1 - x_2) \\ \quad + \mu_2 (x_1 - x_2)^3 + u_1 \\ m_2 \ddot{x}_2 = -\delta_2 \dot{x}_2 - k_2 (x_2 - x_1) \\ \quad + \mu_2 (x_2 - x_1)^3 + u_2 \end{cases} \tag{21}$$

2) Proposed identification and feedback stabilizing control using polynomial structures: The proposed nonlinear discrete polynomial structure that describes perfectly our system is as follow, the sampling time $T_e = 0.01\, s$ and the initial conditions of the state variables
$X_k(0) = \begin{pmatrix} 0.7 & 0 & 0.1 & 0 \end{pmatrix}^T$, with $x_{1,k}$ displacement of the first mass, $\Omega_{1,k}$ velocity of the first mass, $x_{2,k}$ displacement of the second mass and $\Omega_{2,k}$ velocity of the second mass:

$$X_{k+1} = A_1 X_k + A_2\, X_k^{[2]} + \left(B_0 + B_1 X_k\right) U_k \tag{22}$$

with:

$$X_k = \begin{pmatrix} x_{1,k} \\ \Omega_{1,k} \\ x_{2,k} \\ \Omega_{2,k} \end{pmatrix},\ U_k = \begin{pmatrix} u_{1,k} \\ u_{2,k} \end{pmatrix},$$

$$A_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix},$$

$$A_2^T = \begin{pmatrix} a_{15}\, a_{25}\, a_{35}\, a_{45} \\ a_{16}\, a_{26}\, a_{36}\, a_{46} \\ a_{17}\, a_{27}\, a_{37}\, a_{47} \\ a_{18}\, a_{28}\, a_{38}\, a_{48} \\ 0_{4\times4} \\ a_{19}\, a_{29}\ \ 0 \ \ \ a_{49} \\ 0_{7\times4} \end{pmatrix}, B_0 = \begin{pmatrix} b_{11}^0 & b_{12}^0 \\ b_{21}^0 & b_{22}^0 \\ b_{31}^0 & b_{32}^0 \\ b_{41}^0 & b_{42}^0 \end{pmatrix}$$

and $B_1 = \begin{pmatrix} b_{11}^1 & b_{12}^1 & b_{13}^1 & b_{14}^1 & b_{11}^2 & b_{12}^2 & b_{13}^2 & b_{14}^2 \\ b_{21}^1 & b_{22}^1 & b_{23}^1 & b_{24}^1 & b_{21}^2 & b_{22}^2 & b_{23}^2 & b_{24}^2 \\ b_{31}^1 & b_{32}^1 & b_{33}^1 & b_{34}^1 & b_{31}^2 & b_{32}^2 & b_{33}^2 & b_{34}^2 \\ b_{41}^1 & b_{42}^1 & b_{43}^1 & b_{44}^1 & b_{41}^2 & b_{42}^2 & b_{43}^2 & b_{44}^2 \end{pmatrix}.$

The performance of the proposed polynomial structure is assessed using the MSE and the VAF indicators, is presented in Table 2.

TABLE II – Performance Indicators

	MSE	VAF %
$x_{1,\,k}$	$1.5833\,10^{-9}$	99.8068
$\Omega_{1,\,k}$	$5.9269\,10^{-13}$	99.9999
$x_{2,k}$	$1.1949\,10^{-9}$	93.7454
$\Omega_{2,\,k}$	$1.7374\,10^{-10}$	99.9608

To stabilize the CMSD system, we consider the following nonlinear control law:

$$U_k = -\,K_1 X_k \,-\, K_2\, X_k^{[2]} \qquad (23)$$

with:

$$K_1 = \begin{pmatrix} 8.2253 & 10.2009 & 0.7640 & -0.0844 \\ 0.9997 & 0.1870 & 8.1414 & 9.2730 \end{pmatrix}$$

$$\text{and}\ \ K_2^T = \begin{pmatrix} -0.2407 & 0.0139 \\ -0.1001 & 0.0322 \\ 0.1504 & -0.0529 \\ 0.0119 & -0.0044 \\ -0.1708 & -0.0004 \\ -0.0791 & 0.0034 \\ 0.0645 & 0 \\ 0 & 0 \\ 0.0003 & 0.0106 \\ -0.0059 & 0.1479 \\ 0.0048 & 0.0004 \\ 0 & 0 \\ 0.0014 & -0.0190 \\ 0.0007 & 0.1685 \\ -0.0005 & 0.0005 \\ 0 & 0 \end{pmatrix}.$$

3) Simulation results: For parameters estimation of CMSD system, we choose the causal signals $u_{1,\,k} = \frac{1}{3}\sin(k\,\pi\,T_e)$ and $u_{2,\,k} = \frac{1}{5}\sin(k\,\pi\,T_e)$, as inputs of the CMSD system.

The responses of real and estimated state variables $x_{1,k}$ and $x_{2,k}$, as well as, the errors are presented from Figures 2 and 3, respectively.

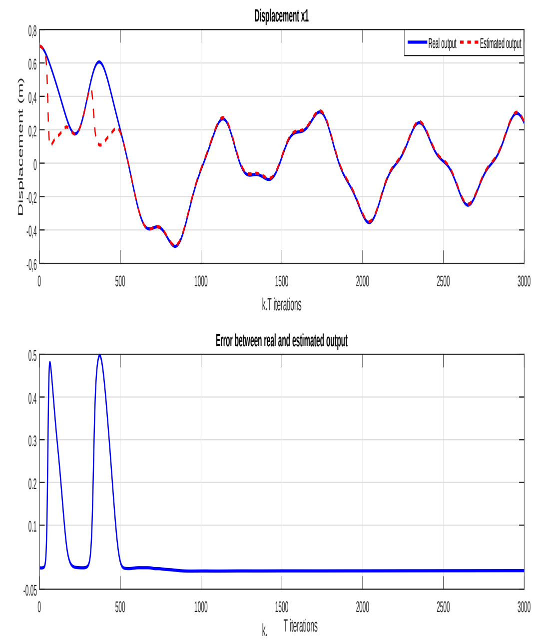

Fig. 2 – Displacement of the first mass $x_{1,k}$ in the open-loop

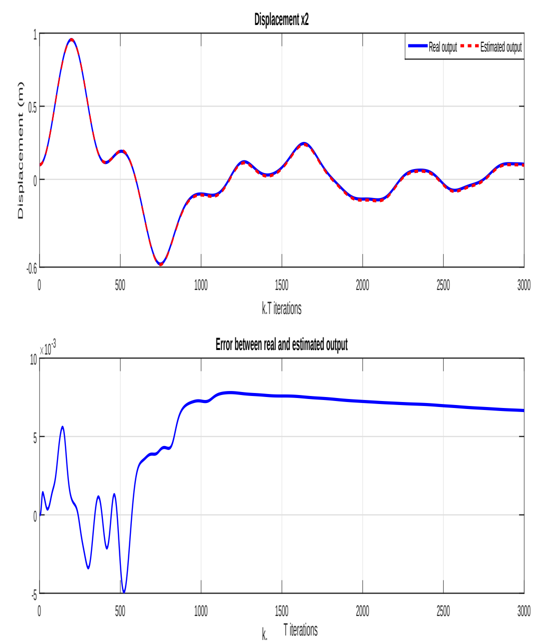

Fig. 3 – Displacement of the second mass $x_{2,k}$ in the open-loop

Figure 4 shows the control signals $u_{1,k}$ and $u_{2,k}$. The responses of the state variables $x_{1,k}$ and $x_{2,k}$ of the CMSD system using nonlinear feedback stabilizing control technique, equation 23, are depicted in Figure 5.

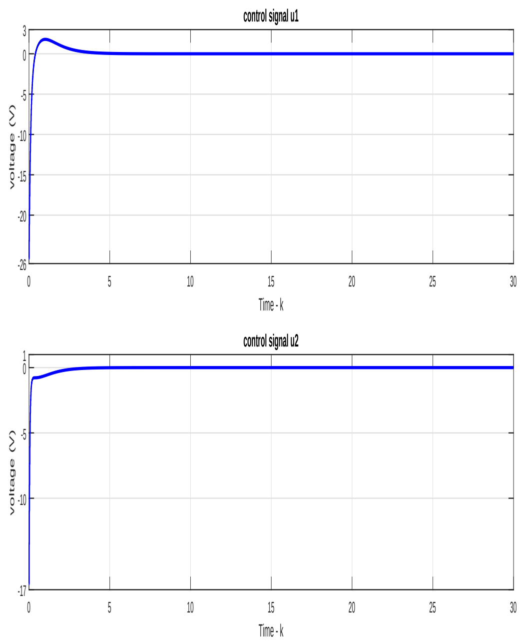

Fig. 4 – Control signals $u_{1,k}$ and $u_{2,k}$

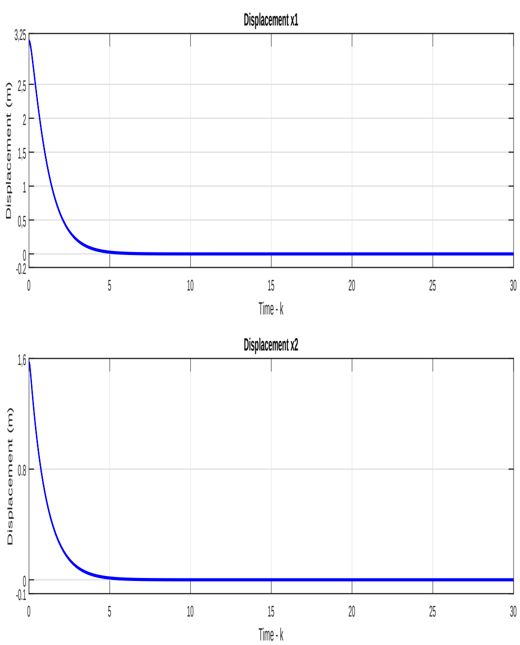

Fig. 5 – Closed-loop displacements $x_{1,k}$ and $x_{2,k}$ evolution

V. Discussion

The main concern of the paper was to determine suitable nonlinear discrete polynomial structure of complex systems, which allowed to design a feedback stabilizing control law.

As can be seen from Figures 2 and 3, the identified outputs tracks the behavior of the real ones perfectly. The modeling errors range of $x_{1,k}$ and $x_{2,k}$ are from -0.05 to 0.5 and -0.001 to 0.005, respectively. As well as, the indicator performance values given in Table 2, the elaborate model applied to the CMSD system can achieve a sufficiently high modeling accuracy.

The convergence of the nonlinear discrete polynomial model parameters values obtained using the RLS algorithm is presented, in Appendix A, Table 3. Indeed, Figure 4 shows that by applying the proposed structure to design feedback gains based on Kronecker power, suitable inputs can be produced for CMSD system that make state variables track equilibrium point rapidly, as given in Figure 5.

VI. Conclusion

A nonlinear discrete polynomial structure has been elaborated. RLS algorithm has been used for the parameters estimation. The polynomial structure allowed to design a feedback stabilizing control law based on Kronecker power for complex systems. The proposed structure has been applied successfully to model and stabilize CMSD system.

Simulation results demonstrate that the identified model has allowed to elaborate a feedback stabilizing control law, which had provided a satisfactory performance in stabilizing the CMSD system at the equilibrium points.

Appendix A

TABLE III – Polynomial Structure Parameters Values

Iterations	$k = 1000$	$k = 2500$	$k = 3000$
$a_{11,k}$	0.8964	0.9889	0.9999
$a_{12,k}$	0.0064	0.0094	0.0100
$a_{13,k}$	$5.0711\,10^{-5}$	$5.0647\,10^{-5}$	$5.0668\,10^{-5}$
$a_{14,k}$	$2.4375\,10^{-7}$	$3.6931\,10^{-7}$	$5.1578\,10^{-7}$
$a_{15,k}$	$6.9419\,10^{-6}$	$6.9371\,10^{-6}$	$7.0247\,10^{-6}$
$a_{16,k}$	$3.7207\,10^{-7}$	$6.9149\,10^{-7}$	$1.0616\,10^{-6}$
$a_{17,k}$	$-3.8943\,10^{-6}$	$-3.9178\,10^{-6}$	$-4.1106\,10^{-6}$
$a_{18,k}$	$-4.4732\,10^{-7}$	$-4.6420\,10^{-7}$	$-4.9244\,10^{-7}$
$a_{19,k}$	$-2.0127\,10^{-6}$	$-1.9510\,10^{-6}$	$-1.8830\,10^{-6}$
$a_{21,k}$	-0.0116	-0.0128	-0.0136
$a_{22,k}$	0.9858	0.9978	0.9988
$a_{23,k}$	0.0088	0.0096	0.0099
$a_{24,k}$	$5.3704\,10^{-5}$	$9.2341\,10^{-5}$	$1.3632\,10^{-4}$
$a_{25,k}$	$9.9317\,10^{-4}$	0.0010	0.0010
$a_{26,k}$	$1.3510\,10^{-4}$	$2.4317\,10^{-4}$	$3.6300\,10^{-4}$
$a_{27,k}$	$-9.2512\,10^{-4}$	$-9.3906\,10^{-4}$	$-9.8158\,10^{-4}$
$a_{28,k}$	$-1.0911\,10^{-4}$	$-1.1247\,10^{-4}$	$-1.1889\,10^{-4}$
$a_{29,k}$	$-1.5265\,10^{-4}$	$-1.4477\,10^{-4}$	$-1.3337\,10^{-4}$
$a_{31,k}$	$9.9534\,10^{-5}$	$9.9533\,10^{-5}$	$9.9535\,10^{-5}$
$a_{32,k}$	$7.7758\,10^{-8}$	$1.1050\,10^{-7}$	$1.2859\,10^{-7}$
$a_{33,k}$	0.9889	0.9987	0.9999
$a_{34,k}$	0.005	0.008	0.01
$a_{35,k}$	$-4.8452\,10^{-6}$	$-4.8423\,10^{-6}$	$-4.8277\,10^{-6}$
$a_{36,k}$	$-3.5350\,10^{-6}$	$-3.4305\,10^{-6}$	$-3.2950\,10^{-6}$
$a_{37,k}$	$5.4944\,10^{-6}$	$5.4922\,10^{-6}$	$5.5046\,10^{-6}$
$a_{38,k}$	$-7.2677\,10^{-7}$	$-6.1974\,10^{-7}$	$-5.3975\,10^{-7}$
$a_{41,k}$	0.0186	0.0190	0.0196
$a_{42,k}$	$5.7104\,10^{-5}$	$5.6609\,10^{-5}$	$5.5627\,10^{-5}$
$a_{43,k}$	-0.0184	-0.0191	-0.0194
$a_{44,k}$	0.9948	0.9954	0.9958
$a_{45,k}$	$-3.0969\,10^{-4}$	$-3.3269\,10^{-4}$	$-3.3269\,10^{-4}$
$a_{46,k}$	$-2.2553\,10^{-4}$	$-2.4861\,10^{-4}$	$-2.8341\,10^{-4}$
$a_{47,k}$	0.0008	0.0010	0.0011
$a_{48,k}$	$9.8465\,10^{-4}$	$1.0059\,10^{-5}$	$8.9020\,10^{-5}$
$a_{49,k}$	$-5.6627\,10^{-4}$	$-5.5898\,10^{-4}$	$-5.3928\,10^{-4}$

Iterations	$k = 1000$	$k = 2500$	$k = 3000$
$b^{0}_{11,k}$	$4.9719\,10^{-5}$	$4.9751\,10^{-5}$	$4.9796\,10^{-5}$
$b^{0}_{12,k}$	$-1.9145\,10^{-7}$	$-1.7296\,10^{-7}$	$-1.4859\,10^{-7}$
$b^{0}_{21,k}$	0.005	0.008	0.01
$b^{0}_{22,k}$	$3.2638\,10^{-4}$	$3.2440\,10^{-4}$	$3.2376\,10^{-4}$
$b^{0}_{31,k}$	$7.3450\,10^{-8}$	$4.6762\,10^{-8}$	$2.2880\,10^{-8}$
$b^{0}_{32,k}$	$1.0049\,10^{-4}$	$1.0046\,10^{-4}$	$1.0042\,10^{-4}$
$b^{0}_{41,k}$	$-2.2969\,10^{-5}$	$-2.9023\,10^{-5}$	$-3.5136\,10^{-5}$
$b^{0}_{42,k}$	0.0190	0.0194	0.0198
$b^{1}_{11,k}$	$1.4001\,10^{-6}$	$1.2392\,10^{-6}$	$1.1519\,10^{-6}$
$b^{1}_{12,k}$	$6.3145\,10^{-6}$	$6.4188\,10^{-6}$	$6.5700\,10^{-6}$
$b^{1}_{13,k}$	$4.2077\,10^{-7}$	$3.0541\,10^{-7}$	$3.0630\,10^{-7}$
$b^{1}_{21,k}$	$1.8451\,10^{-4}$	$1.6802\,10^{-4}$	$1.6751\,10^{-4}$
$b^{1}_{22,k}$	$9.2558\,10^{-5}$	$8.0139\,10^{-5}$	$7.7278\,10^{-5}$
$b^{1}_{23,k}$	$-8.0191\,10^{-5}$	$-6.8981\,10^{-5}$	$-6.3339\,10^{-5}$
$b^{1}_{31,k}$	10^{-6}	$2\,10^{-6}$	$2\,10^{-6}$
$b^{1}_{32,k}$	$3\,10^{-7}$	$4\,10^{-7}$	$5\,10^{-7}$
$b^{1}_{41,k}$	$3.8\,10^{-4}$	$4.2\,10^{-4}$	$4.4\,10^{-4}$
$b^{1}_{42,k}$	$6.4\,10^{-5}$	$7.2\,10^{-5}$	$8.2\,10^{-5}$
$b^{2}_{11,k}$	$-6.17\,10^{-6}$	$-5.63\,10^{-6}$	$-5.64\,10^{-6}$
$b^{2}_{12,k}$	$-6.17\,10^{-6}$	$-5.63\,10^{-6}$	$-5.47\,10^{-6}$
$b^{2}_{13,k}$	$5.09\,10^{-8}$	$1.42\,10^{-7}$	$1.44\,10^{-7}$
$b^{2}_{21,k}$	$-1\,10^{-3}$	$-1.18\,10^{-3}$	$-1.3\,10^{-3}$
$b^{2}_{22,k}$	$-2\,10^{-4}$	$-3\,10^{-4}$	$-4\,10^{-4}$
$b^{2}_{23,k}$	$1.54\,10^{-5}$	$1.86\,10^{-5}$	$1.82\,10^{-5}$
$b^{2}_{31,k}$	$3.82\,10^{-6}$	$4.3\,10^{-6}$	$4.45\,10^{-6}$
$b^{2}_{32,k}$	$6.86\,10^{-7}$	$8.46\,10^{-7}$	$1\,10^{-6}$
$b^{2}_{41,k}$	$6\,10^{-4}$	$7.5\,10^{-4}$	$9\,10^{-4}$
$b^{2}_{42,k}$	$0.5\,10^{-4}$	$1.5\,10^{-4}$	$2\,10^{-4}$

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