On the Sampling and the Performance Comparison of Controlled LTI Systems

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***Abstract*—In this paper, the impact of the discretization techniques and the sampling time, on the finite-time stabilization of sampled-data controlled Linear Time Invariant (LTI) systems, is investigated. To stabilize the process in finite time, a discrete-time feedback dead-beat controller is designed for the sampled-data system. Checkable conditions on the approximate discrete-time plant model and the associated controller that guarantee the finite-time stabilization of the exact model are developed. The trade-off between the discretization technique, the sampling time and the desired performances is illustrated and discussed. Results are presented through a case study.**

Keywords—Sampled-data systems; discretization; finite time stabilization; dead-beat control

I. Introduction

Most real systems evolve naturally in continuous time, but, nowadays, modern control strategies are typically implemented through digital devices to meet high-demanding control performance specifications. For these purposes, different discretization schemes and numerical approximation techniques are developed [1]-[4]. The most cited methods include the Zero Order Hold (ZOH) technique, which is an exact sampling-data representation of the original continuous system, and numerical approximations techniques [4], [5]. The ZOH is placed at the input of the considered process to hold the input signal constant until the next sample becomes available. However, very often exact solutions of the differential equations process are not available. In consequence, numerical approximation approaches turn out to be essential to yield accurate approximations of the real solutions [5], Euler approximation methods (forward and backward) are basic approaches of numerical integration [4], [5]. The Euler rule for discrete approximation of integral functions between two sampling instant gives an approximate area of a rectangle whose base is the sampling interval and whose height is the value of the function at the lower limit (forward) or final limit (backward Euler). This simple and easy to implement technique became a popular digital implementation method. The stability conditions and convergence of the Euler techniques have been developed for linear systems [3] and for some classes of nonlinear systems [6]. The main disadvantage of the Euler's techniques is that either they overestimate or underestimate the integral. Tustin (bilinear, trapezoidal) approximation [7], [8] overcomes these disadvantages by taking the average of the limiting values of the integrand. As such, it treats the area between the two integration limits as trapezium.

The problems of control design and performances analysis for linear and nonlinear dynamical systems [9]-[18] are still important nowadays. Three techniques are known in the literature for the digital controllers construction of continuous-time systems: 1) Design of a continuous-time controller and, then, its discretization; 2) Discretization of the plant and construction of a discrete-time controller on the basis of the associated sampled-data model [19]; and 3) Direct digital controller design based on a continuous-time plant model without approximations [19]. Cited techniques show acceptable performances when the sampling is fast. But, the discrete construction controller method (b) does not need a fast sampling to maintain stability as it utilizes an approximation of the process ignoring the inter-sample system behavior.

In practical engineering processes, the increase need of time performance criteria and exact time specifications of the dynamics behaviors has led to the development concept of finite-time convergence stability and stabilization [20]. Considered in the literature of dead-beat control and optimality, the capacity to force a dynamic control system to reach a specified target in a finite time called settling time, represents the main merit of the finite-time control. Finite-time stabilization techniques have attracted a great deal of attention and have become a heated research issue in control systems theory [21]-[24]. Early works on the topic developed relevant finite-time stabilization approaches for different classes of linear and nonlinear discrete-time systems. Researchers have investigated the finite-time stabilization of discrete time linear time-varying systems in [25], subject to disturbances in [26], with time-varying delay in [27]and uncertain and subject to exogenous disturbances in [28]. Finite-time stabilization issues of nonlinear plants have been investigated. Systems which can be represented by affine fuzzy systems were considered in [24], the class Lur'e type systems in [5] and uncertain systems in [29], [30]. Although the encouraging works in the field, investigations about the effect of the sampling technique on the system performance in term of stability and finite-time stabilization were not included.

In this note we discuss important issues when selecting the sampled-data description in the context of a dead-beat control applied to LTI systems. The discussion is based on an example of a third order system that, from our point of view, can be understood in a common framework to select discrete-time models and synthesize dead-beat controllers. The treatment is limited for simplicity to the linear case but the extension to the nonlinear case is possible, at least for some class of Lure-type systems, by considering sector bounded nonlinearities [23].

The remainder of this work is organized as follows. In Section II, the problem setup to be considered is introduced and previous results concerning the stability and the finite-time stabilization are recalled. In Section III, a discussion of the influence of the discretization techniques and the sampling times on the system performances is developed through a third order case study. Concluding remarks are drawn in Section IV.

II. Problem Statement and Preliminaries

In this section, the sampled-data system under consideration is introduced and the problem formally stated.

A. System Description and Discretization

Let's consider the controlled sampled-data system S of Fig. 1, in which S_C denotes a continuous Linear Time Invariant (LTI) plant. Blocs (A-D), (DTC), and (D-A) designate the zero-order hold, the discrete-time controller and the ideal sampler, respectively, synchronized by the same sampling time T. We suppose that S_C can be described by the n^{th} following differential equation:

$$S_C : y^{(n)} + \sum_{i=1}^{n} a_i^c y^{(n-i)} = \sum_{j=0}^{n-1} b_j^c u^{(n-1-j)} \quad (1)$$

and in the state space description by

$$S_C : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \quad (2)$$

$u(t) \in \mathbb{R}$ denotes the control signal delivered to the plant, $y(t) \in \mathbb{R}$ the plant's measured output, $x(t) = (y, y^{(1)}, \ldots, y^{(n-1)})^T \in \mathbb{R}^n$ the state vector and t the time. a_i^c and b_j^c are constant parameters for $i = 1, 2, \cdots, n$ and $j = 0, 1, \cdots, n-1$. A, B and C are known matrices of appropriate dimensions.

It is desired to develop a state feedback control to the introduced continuous-time system in a discrete-time approach by sampling the continuous plant and applying a discrete-time controller. Using the sampler and ZOH, the continuous plant (2) is discretized to the following exact sampled-data representation

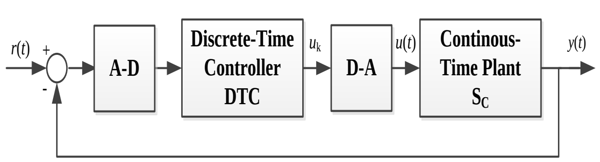

Fig. 1. Sampled-data controlled system.

$$(S_{SD}) : \begin{cases} x_{k+1} = Fx_k + Gu_k \\ y_k = Cx_k \end{cases} \quad (3)$$

Applying, now, the discrete-time state feedback control

$$u_k = -\Lambda x_k + Lr_k \quad (4)$$

the closed-loop controlled system becomes

$$(S) : \begin{cases} x_{k+1} = Mx_k + GLr_k \\ y_k = Cx_k \end{cases} \quad (5)$$

With

$$M = F - G\Lambda \quad (6)$$

x_k, y_k, u_k and r_k define, respectively, $x(kT)$, $y(kT)$, $u(kT)$ and $r(kT)$; $k \in \mathbb{N}$. We assume that the state measurements x_k are available at sampling instants kT and the pair (F, G) is controllable. $F \in \mathbb{R}^{n \times n}$ and $G \in \mathbb{R}$ are the discrete system matrices defined, respectively, by [4]

$$F = e^{AT}, \quad G = \int_0^T e^{At} B dt \quad (7)$$

$\Lambda = [\lambda_j] \in \mathbb{R}^n$, $j = 0, 1, \cdots, n-1$ and $L = [\gamma] \in \mathbb{R}$ are the static gains controller.

Finally, we suppose that the controlled sampled-data system (5)-(6) can be described by the recursive input/output scalar equation, as

$$(S) : y_{k+n} + \sum_{i=1}^{n} a_i y_{k+n-i} = \sum_{j=0}^{n-1} b_j r_{k+n-1-j} \quad (8)$$

ZOH discretization approach leads to an exact sampled-data model S_{SD}; the continuous-time output of (2) is exactly recovered at the sampling instants, i.e., $y_k = y(kT)$. But, in many cases, analytical integration may be impossible or infeasible, in particular for nonlinear systems. Numerical integration techniques become essential to yield accurate approximations of the actual solutions. In that respect, forward Euler and Tustin are basic and well used approximations approaches. An approximate discrete-time state space model of (2) can be given by

$$(\hat{S}_{SD}) : \begin{cases} x_{k+1} = \hat{F}x_k + \hat{G}u_k \\ y_k = Cx_k \end{cases} \quad (9)$$

where, for forward Euler approximation, matrices $\hat{F}$ and $\hat{G}$ are defined, respectively, by [3], [4]

$$\hat{F} = I + TA, \quad \hat{G} = TB \quad (10)$$

and for Tustin approximation, respectively, by [19], [31]

$$\hat{F} = \left(\frac{I}{T} - \frac{A}{2}\right)^{-1} \left(\frac{I}{T} + \frac{A}{2}\right), \quad \hat{G} = \left(\frac{I}{T} - \frac{A}{2}\right)^{-1} B \quad (11)$$

The evaluation of the exponential and integral matrices (7) for the exact discretization technique, and the matrix inversion (11) for the Tustin approximation, are generally time-consuming and may necessitate a high speed processor for real-time implementations, especially for large scale systems.

Besides, the forward Euler approximation technique is a simpler and less costly representation.

B. Finite-Time Stabilization and Problem Statement

Stability property of the discretized LTI closed loop system (5)-(6) depends on the state matrix M and the sampling time choice. Let us denote by $\sigma(M)$ the spectral radius, i.e., $\sigma(M)=Max\{|\lambda|:\lambda$ is an eigenvalue of M $(n\times n)$ matrix$\}$.

Lemma 1. [32] If $\sigma(M)<1$, the eigenvalues of the matrix M are located within the unit circle, then the system (5)-(6) is asymptotically stable.

Considered as an advanced control design technique, dead-beat control [33], [34] is developed in the context of finite time stabilization and finite settling time, which aims to perfectly tracking a step reference in a finite number of sampling periods.

Definition 1. [35] A stabilizing controller of systems are said to be a dead-beat controller if the tracking output $y(kT)$ settles down to zero in a finite number of steps $k=N_d$ and $y(kT)=0$, $\forall\ k\geq N_d$; N_d is the settling step.

Definition 2. [36] Consider a continuously working system. Then, the finite number of control steps is such a finite number of values n of the sequence $U=\{u(k_0),\, u(k_0+1),\, \ldots,\, u(k_0+n-1)\}$ by which the system is transferred from an arbitrary initial state $x(k_0)\neq 0$ to the final state $x(k_0+n)=0$, $\forall k_0\in$ □ .

Now, let consider the discretized system described by (8) such that for $\forall i=1,2,\cdots n$, a_i are expressed in terms of λ_j, $j=0,1,\ldots,n-1$, and T.

Lemma 2. [37] The n^{th} linear discrete-time system (8) is said to be stabilizable in n sampling time, if the gains λ_j, $j=0,\cdots,n-1$, and T, are synthesized, such that y_{k+n} settles down to zero in n steps, that's equivalent to setting $(F-G\Lambda)^n=0$ ($(F-G\Lambda)$ is nilpotent with the index n), or equivalently, the $a_i\equiv 0$ for $i=1,2,\cdots n$.

This note investigates the influence of the discretization method in the controlled system properties. The work is a continuation of the previous paper [23] in which the effects of a ZOH discretization on the stability and stabilization properties of linear and nonlinear Lure-type systems are considered. The aim of this paper is to investigate more discretization techniques, mainly, the forward Euler and the Tustin approximations, and study the impact of a numerical approximation on the control system (5) on the system properties in term of stability and finite-time stability.

III. Case Study

In this section, we develop sufficient finite-time stability conditions of a controlled sampled-data third-order system via 1) the exact solution; and 2) the discrete-time model approximations. Based on the continuous-time plant model, our main results specify checkable conditions ensuring that a finite-time control stabilizing the approximate model would also stabilizes the exact model in finite-time. These conditions can be used as guidelines for controller design based on approximate nonlinear models.

The main results described in this case study focus on the at sample response.

A. Studied Third Order System Description and Discretization

Consider the third order continuous LTI system configured as in Fig. 1, with the following plant transfer function:

$$S_c : F(s)=\frac{k_s}{s(1+\tau_1 s)(1+\tau_2 s)} \tag{12}$$

leading to the following state space controllable form:

$$A=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\dfrac{1}{\tau_1\tau_2} & -\dfrac{\tau_1+\tau_2}{\tau_1\tau_2} \end{pmatrix},\ B=\begin{pmatrix} 0 \\ 0 \\ \dfrac{k_s}{\tau_1\tau_2} \end{pmatrix},\ C=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^T \tag{13}$$

k_s is the static gain and τ_1 and τ_2 the constant times of the continuous-time process. The developed sampled-data models of (13) based on the ZOH technique, forward Euler and the Tustin approximations, are $S_{SD}:\left(F,G,C\right)$, $\hat{S}^e_{SD}:\left(\hat{F}^e,\hat{G}^e,C\right)$ and $\hat{S}^t_{SD}\left(\hat{F}^t,\hat{G}^t,C\right)$, respectively, such that

$$F=\begin{pmatrix} 1 & \dfrac{\tau_1^2(1-d_1)-\tau_2^2(1-d_2)}{\tau_1-\tau_2} & \dfrac{\tau_1\tau_2\left(\tau_1(1-d_1)-\tau_2(1-d_2)\right)}{\tau_1-\tau_2} \\ 0 & \dfrac{\tau_1 d_1-\tau_2 d_2}{\tau_1-\tau_2} & \dfrac{\tau_1\tau_2(d_1-d_2)}{\tau_1-\tau_2} \\ 0 & \dfrac{d_2-d_2}{\tau_1-\tau_2} & \dfrac{\tau_1 D_2-\tau_2 d_1}{\tau_1-\tau_2} \end{pmatrix}$$

$$G=k_s\begin{pmatrix} T-(\tau_1+\tau_2)+\dfrac{\tau_1^2 d_1-\tau_2^2 d_2}{\tau_1-\tau_2} \\ 1-\dfrac{\tau_1 d_1-\tau_2 d_2}{\tau_1-\tau_2} \\ \dfrac{d_1-d_2}{\tau_1-\tau_2} \end{pmatrix} \tag{14}$$

$d_1^{-1}=e^{\frac{T}{\tau_1}}$ and $d_2^{-1}=e^{\frac{T}{\tau_2}}$,

$$\hat{F}^e=\begin{pmatrix} 1 & T & 0 \\ 0 & 1 & T \\ 0 & -\dfrac{T}{\tau_1\tau_2} & 1-\dfrac{(\tau_1+\tau_2)T}{\tau_1\tau_2} \end{pmatrix},\quad \hat{G}^e=\begin{pmatrix} 0 \\ 0 \\ \dfrac{Tk_s}{\tau_1\tau_2} \end{pmatrix} \tag{15}$$

and

$$\hat{F}^t = \begin{pmatrix} 1 & \frac{2T\left(T\left(\tau_1+\tau_2\right)+2\tau_1\tau_2\right)}{\left(T+2\tau_1\right)\left(T+2\tau_2\right)} & \frac{2T^2\tau_1\tau_2}{\left(T+2\tau_1\right)\left(T+2\tau_2\right)} \\ 0 & \frac{2T\left(\tau_1+\tau_2\right)+4\tau_1\tau_2-T^2}{\left(T+2\tau_1\right)\left(T+2\tau_2\right)} & \frac{4T\tau_1\tau_2}{\left(T+2\tau_1\right)\left(T+2\tau_2\right)} \\ 0 & -\frac{4T}{\left(T+2\tau_1\right)\left(T+2\tau_2\right)} & -\frac{2T\left(\tau_1+\tau_2\right)-4\tau_1\tau_2+T^2}{\left(T+2\tau_1\right)\left(T+2\tau_2\right)} \end{pmatrix} \quad (16)$$

$$\hat{G}^t = \begin{pmatrix} \frac{k_sT^3}{\left(T+2\tau_1\right)\left(T+2\tau_2\right)} \\ \frac{2k_sT^2}{\left(T+2\tau_1\right)\left(T+2\tau_2\right)} \\ \frac{4k_sT}{\left(T+2\tau_1\right)\left(T+2\tau_2\right)} \end{pmatrix}$$

Applying a state feedback controller (4), with $\Lambda = \left(\lambda_0 \quad \lambda_1 \quad \lambda_2\right)$, the resulting free closed-loop system based on the zero order hold (exact) discretization can be expressed by the recursive input/output (8) where the coefficients a_i $\left(i=1,2,3\right)$ (A.1)-(A.3) are developed in the Appendix. Now, based on the model approximation, the free controlled system based on the forward Euler technique is such that

$$\hat{S}^e : \; y_{k+3} + \sum_{i=1}^{3} \hat{a}_i^e y_{k+3-i} = 0 \quad (17)$$

with

$$\begin{cases} \hat{a}_1^e = \dfrac{T}{\tau_1\tau_2} \times \left(\tau_1+\tau_2+k_s\lambda_2\right) - 3 \\ \hat{a}_2^e = \dfrac{T}{\tau_1\tau_2} \times \left(\left(1+k_s\lambda_1\right)T - 2k_s\lambda_2 - 2(\tau_1+\tau_2)\right) + 3 \\ \hat{a}_3^e = \dfrac{T}{\tau_1\tau_2} \times \left(\lambda_0 k_s T^2 - \left(1+k_s\lambda_1\right)T + \lambda_2 k_s + \tau_1 + \tau_2\right) - 1 \end{cases} \quad (18)$$

and on the Tustin technique

$$\hat{S}^t : \; y_{k+3} + \sum_{i=1}^{3} \hat{a}_i^t y_{k+3-i} = 0 \quad (19)$$

with

$$\begin{cases} \hat{a}_1^t = \dfrac{\lambda_0 k_s T^3 + \left(2\lambda_1 k_s + 1\right)T^2 + 2\left(2\lambda_2 k_s - \tau_1 - \tau_2\right)T - 12\tau_1\tau_2}{\left(T+2\tau_1\right)\left(T+2\tau_2\right)} \\ \hat{a}_2^t = \dfrac{2\lambda_0 k_s T^3 - T^2 - 2\left(\tau_1+\tau_2+4\lambda_2 k_s\right)T + 12\tau_1\tau_2}{\left(T+2\tau_1\right)\left(T+2\tau_2\right)} \\ \hat{a}_3^t = \dfrac{\lambda_0 k_s T^3 - \left(2\lambda_1 k_s + 1\right)T^2 + 2\left(2\lambda_2 k_s + \tau_1 + \tau_2\right)T - 4\tau_1\tau_2}{\left(T+2\tau_1\right)\left(T+2\tau_2\right)} \end{cases} \quad (20)$$

Next, we study the finite-time stability based on the exact discretization technique and on the approximations methods.

B. Finite-Time Stability

To illustrate the finite-time stability behavior of the controlled sampled data developed models (8) with (A.1)-(A.3), (17)-(18) and (19)-(20), we consider the process (12) parameters such that $\tau_1 = 0,05s$, $\tau_2 = 0,1s$ and $k_s = 5,89$. The dead-beat controller's output is calculated according to Lemma 2.

1) Control Synthesis based on the Exact Discretion

Based on the ZOH discretization, synthesized controller u_k gains are:

$$\lambda_0 = 1\,; \quad \lambda_1 = 1,45\times10^{-1}\,; \quad \lambda_2 = 4,8\times10^{-3} \quad (21)$$

for a sampling time $T_{db} = 0,2s$ [23]. In order to illustrate the developed controller performances applied to (a) the sampled-data system using a sampler and zero order hold (8) with (A.1)-(A.3), (b) the forward Euler based approximation system (17)-(18) and (c) the Tustin based approximation system (19)-(20), conditions relating stability domains and sampling periods (formulation based on the Jury criteria) are presented in Table I. Maximum values of sampling times T^*, T_e^* and T_t^* (corresponding to ZOH, forward Euler and Tustin discretizations, respectively) that can be simulated maintaining the stability of the sampled data closed-loop system, are calculated. While T_t^* is close to T^*, we note that $T_e^* \ll T^*$; the closed loop-system (17)-(18) is stabilizable for very small time steps with $\Omega^e \subset \Omega^t$ and $\Omega \subset \Omega^t$. Stability conditions introduced in Table I. are tested numerically. The ZOH and Tustin discretization-based techniques are applied with five (05) values of discretization steps ($0,1$; $0,12$; $0,15$; $0,17$ and $0,2$). Forward Euler discretization-based technique is applied with three (03) sampling periods ($0,04$; $0,06$; $0,07$). Simulations are carried on for the same initial conditions $x\left(0\right) = 10^{-1} \times \left[1 \quad 0,11 \quad -0,73\right]^T$. The discrete dynamics of the controlled system based, ZOH, forward Euler and Tustin schemes, are shown in Fig. 2, 3 and 4, respectively. From Fig. 2, it is easy to check that (a) the controlled system based ZOH discretization is asymptotically stable for selected numerical sampling times verifying $T \in \Omega$ and (b) the developed dead-beat control with (21) ensures a transient behaviour elimination in three (03) sampling periods for $T_{db} = 0,2$ with a settling time $t_s = 0,6s$. When the sampling period decreases, the models exhibit a finite-time stability convergence to $t_s = 0,6s$ with a number of steps $m > 3$. Based on simulation results shown in Fig. 3 and 4, forward Euler and Tustin approximations demonstrate, respectively, a stability convergence to the origin $x = 0$, for the developed control (21) for $T \in \Omega^e$ and $T \in \Omega^t$, respectively. Now, comparing the finite time stability performance of the exact discretization ZOH, and approximate based Euler and Tustin techniques, we notice that for the different sampling times (a) the control (21) stabilizes controlled systems based Euler and Tustin approximations, (b) no settling time t_1 ; $t_1 < t_s = 0,6s$ is obtained, (c) Tustin approximation fails to exhibit a three-sampling time convergence to the origin with T_{db} but, displays a finite time convergence with a settling time $t_s^t = t_s$ for $T = 0{,}12s < T^{db}$ and $T = 0{,}15s < T^{db}$. The performance deteriorates if we increase the sampling time, as shown in Fig. 4(d) Forward Euler approximation is finite-time stable in $t_s^e = 0{,}77s > t_s$, as shown in Fig. 3. The system shows bad finite-time performances, for small sampling periods. The highest sampling periods that guarantee a minimum settling time are summarized in Table II. It is observed that, with a ZOH discretization, the states of the proposed dead-beat controlled system settle down to the steady-state in three sample periods where $T_{db} = 0{,}2s$. For a relatively shorter sampling period $T = 0{,}15s$ and based on a Tustin approximation, the states of the sampled-data controlled system settles down in four sampling periods for $t_s^t = t_s = 0{,}6s$, whereas the states of the conventional forward Euler digitally redesigned sampled-data controlled system has the largest settling time with $t_s^e = 0{,}77s$ and exhibits a Finite-Time Stability (FTS) convergence in eleven sampling periods with $T = 0{,}07s$. The approximation introduced by the forward Euler model affects the desired dead-beat response and creates an unexpected behaviour for relatively small T.

TABLE I. STABILITY DOMAINS

	Zero Order Hold Technique	Forward Euler Approximation	Tustin Approximation
Stability domains	$\Omega : 0{,}005s < T < 0{,}336s$	$\Omega^e : 0s < T < 0{,}099s$	$\Omega^t : 0s < T < 0{,}354s$

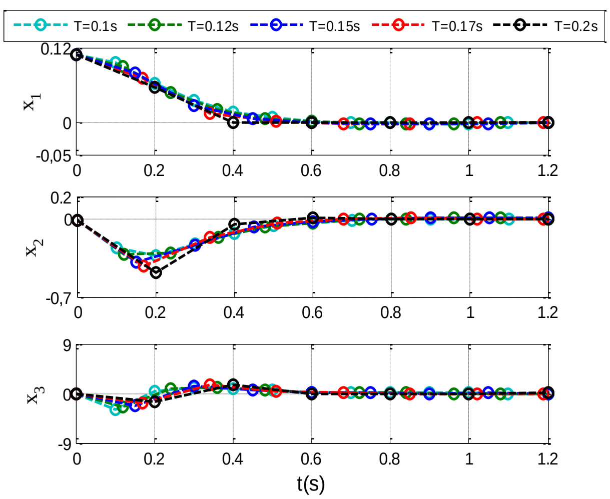

Fig. 2. Sampled-Data Controlled Model Dynamics - ZOH Technique

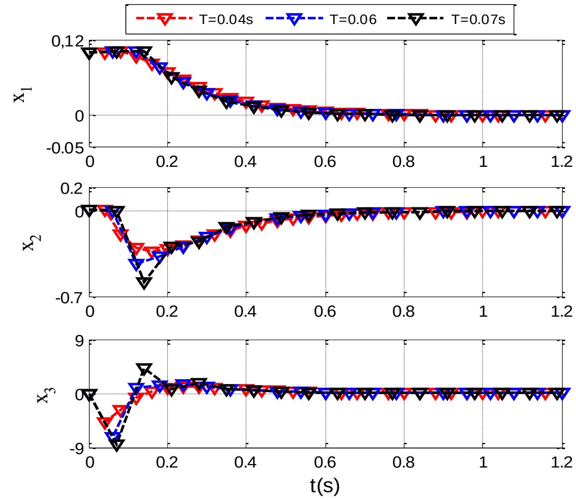

Fig. 3. Sampled-Data Controlled Model Dynamics - Forward Euler Approximation

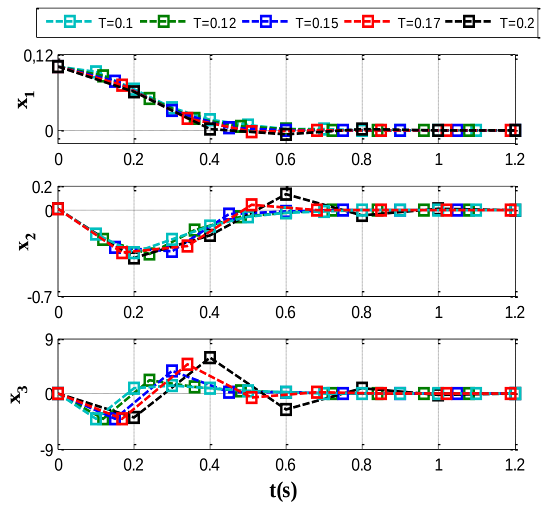

Fig. 4. Sampled-Data Controlled Model Dynamics - Tustin Approximation

TABLE II. FTS COMPARISON OF THE ZOH, FORWARD EULER AND TUSTIN DISCRETIZATIONS

	Sampling Time [s]	Settling Time [s]	Number of Steps
ZOH Technique	$T = 0{,}2$	$t_s = 0{,}6$	3
Forward Euler Approximation	$T = 0{,}07$	$t_e^t = 0{,}77$	11
Tustin Approximation	$T = 0{,}15$	$t_s^t = 0{,}6$	4

2) Controller Synthesis based Tustin Approximate Discretization

Now, the aim is to test if a dead-beat controller which stabilizes a Tustin based approximate discrete-time model also stabilizes in three sampling times the exact discrete model (based on the ZOH) of the plant. For convenience, we use $\left(u_{k,T_2}^{t}\right)$ to refer to the dead-beat controller based Tustin approximation for a sampling time T_2 and $S_t:\left(F_{T_1},u_{k,T_2}^{t,db}\right)$ to the sampled-data model F, based ZOH discretization with a sampling time T_1, for which a stabilizing control $u_{k,T_2}^{t,db}$ is applied. The dead-beat controllers $u_{k,T_2}^{t,db}$ parameters λ_0, λ_1 and λ_2, relevant to the sampling periods $T_2=0{,}1$, $0{,}12$, $0{,}15$, $0{,}17$ and $0{,}2$ are shown in Table III. These parameters values are computed referring to Lemma 2. We note, in regard to the obtained gain parameters, that the λ_0, λ_1 and λ_2 values become quite large as the sampling period becomes smaller. Developed state feedback dead-beat control $u_{k,T_2}^{t,db}$ is, then, applied to the associated exact sampled-data model with $T_1=T_2$. The essential characteristics of the system $S_t:\left(F_{T_1},u_{k,T_2}^{t,db}\right)$ response, obtained with the proposed control are illustrated in the in Fig. 5. As depicted, the controlled sampled-data systems $S_t:\left(F_{T_1},u_{k,T_2}^{t,db}\right)$, for $T_1=T_2=0{,}1$, $0{,}12$, $0{,}15$, $0{,}17$ and $0{,}2$, are finite-time convergent in 5, 5, 5, 5 and 4 steps, respectively. The synthesized dead-beat control $u_{k,T_2}^{t,db}$, cannot stabilize the associated exactly discretized system in $3-$ sampling periods. In order to optimize the system's performance in accordance to the specified objective, we propose to (i) calculate the dead-beat control u_{k,T_1}^{db} based on the ZOH discretization for the sampling period T_1, then, (ii) for obtained value parameters, solve $\hat{a}_i^t=0$, $i=1,2,3$ (20), for T_2. The correspondence between the sampling periods T_1 and T_2 is developed in Table IV. The main point noted from this data is that, considering the set of sampling periods $\Delta=\{0{,}1;\,0{,}12,\,0{,}15,\,0{,}17;0{,}2\}$, T_2 is smaller than T_1. More tests, for T_1 in $[0{,}01\,..\,0{,}3]$, have been carried on. The results are depicted in Fig. 6. Clearly, while, for small sampling periods $T_1\le 0{,}05$, we obtain $T_2=T_1$; for larger sampling periods, $T_2<T_1$. Fig. 7 shows the simulations results. The controlled system performance gets better for new developed $u_{k,T_2}^{t,db}$. The system states are brought to the origin in three steps. By consequence, for large sampling periods, the dead-beat controller based on the Tustin approximation can lead to an *n*-finite-time stabilization of the exact sampled-data *n*-order system by choosing $T_2<T_1$.

TABLE III. DEAD-BEAT CONTROLLER SYNTHESIS BASED ON THE TUSTIN APPROXIMATION

Sampling Period	u_{k,T_2}^{t} **Dead-Beat Controller Parameters**
$T_2=0{,}1$	$\lambda_0=2{,}54,\lambda_1=3{,}39\times10^{-1}$ and $\lambda_2=1{,}06\times10^{-2}$
$T_2=0{,}12$	$\lambda_0=1{,}72,\lambda_1=2{,}45\times10^{-1}$ and $\lambda_2=7{,}9\times10^{-3}$
$T_2=0{,}15$	$\lambda_0=1{,}10,\lambda_1=1{,}60\times10^{-1}$ and $\lambda_2=5{,}1\times10^{-3}$
$T_2=0{,}17$	$\lambda_0=0{,}86,\lambda_1=1{,}23\times10^{-1}$ and $\lambda_2=3{,}8\times10^{-3}$
$T_2=0{,}2$	$\lambda_0=0{,}63,\lambda_1=0{,}84\times10^{-1}$ and $\lambda_2=2{,}1\times10^{-3}$

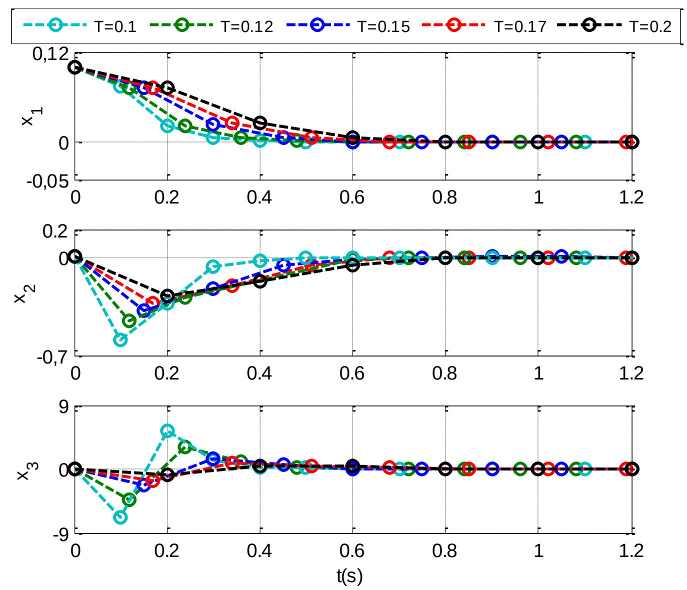

Fig. 5. Controlled Exact sampled-Data Model dynamics – Case 1: Dead-Beat Control based on the Tustin approximation with $T_1=T_2$

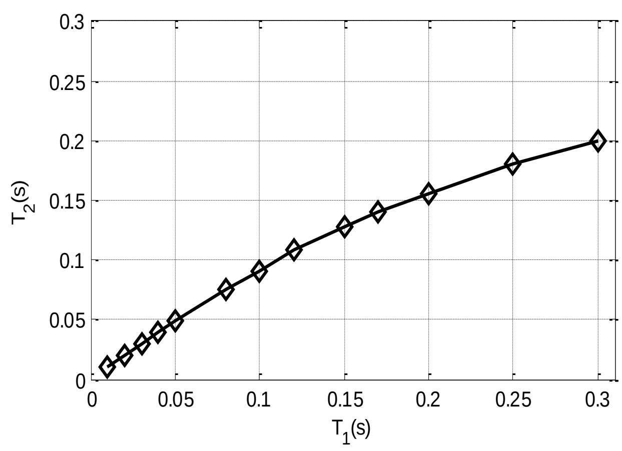

Fig. 6. T_1-T_2 Matching

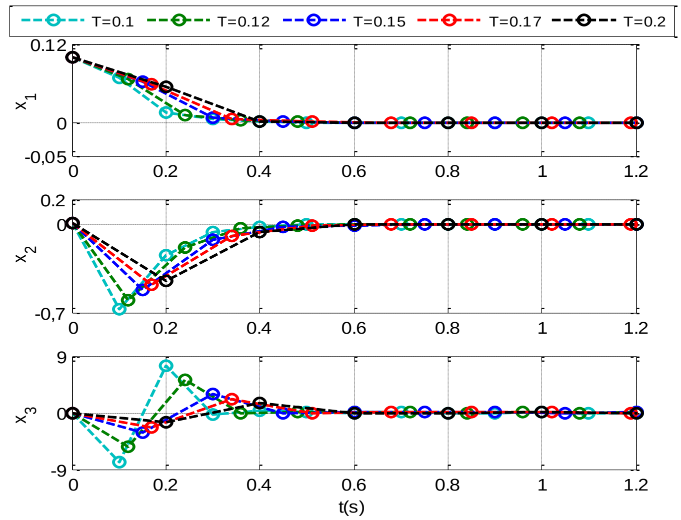

Fig. 7. Controlled Exact sampled-Data Model dynamics – Case 2: Dead-Beat Control based on the Tustin approximation with $T_2 < T_1$

IV. Conclusion

The finite-time stabilization issues of sampled-data linear time invariant systems are discussed. Based on a third-order case study, sufficient conditions ensuring the finite-time stabilization for the exact and the approximate Forward Euler and Tustin approximation models, are derived. Models performances have been compared for some stabilizing sample periods. It has been shown that, while Forward Euler technique outlines a constraining stability domain, Tustin approximation shows better performances. Moreover, it was observed that under some matching conditions, the dead-beat controller based on the Tustin approximation leads to an *n*-finite-time stabilization of the exact sampled-data *n* order system. Developed results can be extended to finite-time stabilization of nonlinear systems as shown in [23].

TABLE IV. Dead-Beat Controller Synthesis Based on the Tustin Approximation with $T_2 < T_1$

T_1	T_2	u^t_{k,T_2} **Dead-Beat Controller Parameters**
$T_1 = 0{,}1$	$T_2 = 0{,}09$	$\lambda_0 = 3{,}10, \lambda_1 = 0{,}37$ and $\lambda_2 = 1{,}12\times10^{-2}$
$T_1 = 0{,}12$	$T_2 = 0{,}108$	$\lambda_0 = 2{,}22, \lambda_1 = 0{,}29$ and $\lambda_2 = 0{,}89\times10^{-2}$
$T_1 = 0{,}15$	$T_2 = 0{,}128$	$\lambda_0 = 1{,}53, \lambda_1 = 0{,}21$ and $\lambda_2 = 0{,}68\times10^{-2}$
$T_1 = 0{,}17$	$T_2 = 0{,}14$	$\lambda_0 = 1{,}26, \lambda_1 = 0{,}18$ and $\lambda_2 = 0{,}58\times10^{-2}$
$T_1 = 0{,}2$	$T_2 = 0{,}155$	$\lambda_0 = 1, \lambda_1 = 0{,}14$ and $\lambda_2 = 0{,}48\times10^{-2}$

Appendix

The free controlled sampled-data model based on a zero order hold discretization of (12) can be given by

$$S:\ y_{k+3} + \sum_{i=1}^{3} a_i\, y_{k+3-i} = 0$$

where the coefficients a_i, $i = (1,2,3)$, are defined, respectively, by

$$a_1 = \left(d_1^{-1}d_2^{-1}(\tau_2-\tau_1)\right)^{-1} \times \begin{pmatrix} k_s\lambda_0 d_2^{-1}\left(d_1^{-1}-d_2^{-1}\right)\tau_1^2 + k_s\lambda_0 d_1^{-1}\left(1-d_2^{-1}\right)\tau_2^2 \\ +\left(d_1^{-1}+d_2^{-1}+d_1^{-1}d_2^{-1}+k_s\lambda_1 d_2^{-1}\left(1-d_1^{-1}\right)-Tk_s\lambda_0 d_1^{-1}d_2^{-1}\right)\tau_1 \\ -\left(d_2^{-1}+d_1^{-1}+d_1^{-1}d_2^{-1}+k_s\lambda_1 d_1^{-1}\left(1-d_2^{-1}\right)-Tk_s\lambda_0 d_1^{-1}d_2^{-1}\right)\tau_2 \\ +k_s\lambda_2\left(d_1^{-1}-d_2^{-1}\right) \end{pmatrix} \quad \text{(A.1)}$$

$$a_2 = k_s\left(\left(d_1^{-1}d_2^{-1}\right)(\tau_2-\tau_1)\right)^{-1} \times \begin{pmatrix} \lambda_0\tau_1^2\left(1-d_1^{-1}+d_2^{-1}-d_1^{-1}d_2^{-1}\right) \\ -\lambda_0\tau_2^2\left(1+d_1^{-1}-d_2^{-1}-d_1^{-1}d_2^{-1}\right) \\ +\tau_1\begin{pmatrix}\lambda_1\left(d_1^{-1}-d_2^{-1}+d_1^{-1}d_2^{-1}-1\right)+ \\ T\lambda_0\left(d_1^{-1}+d_2^{-1}\right)-d_1^{-1}-d_2^{-1}-1\end{pmatrix} \\ +\tau_2\begin{pmatrix}\lambda_1\left(d_1^{-1}-d_2^{-1}-d_1^{-1}d_2^{-1}+1\right)- \\ T\lambda_0\left(d_1^{-1}+d_2^{-1}\right)+d_1^{-1}+d_2^{-1}+1\end{pmatrix}-2\lambda_2\left(d_1^{-1}-d_2^{-1}\right) \end{pmatrix} \quad \text{(A.2)}$$

$$a_3 = d_2 d_1^{-1}(\tau_2-\tau_1)^{-1} \times \begin{pmatrix} \tau_1-\tau_2 \\ -k_s\lambda_0\left(\tau_1^2-\tau_2^2-\tau_1^2 d_1^{-1}+\tau_2^2 d_2^{-1}\right) \\ +k_s\lambda_1\left(\tau_1-\tau_2-\tau_1 d_1^{-1}+\tau_2 d_2^{-1}\right) \\ +k_s\lambda_2\left(d_1^{-1}-d_2^{-1}\right)-Tk_s\lambda_0(\tau_1-\tau_2) \end{pmatrix} \quad \text{(A.3)}$$

with $d_1^{-1} = e^{\frac{T}{\tau_1}}$ and $d_2^{-1} = e^{\frac{T}{\tau_2}}$

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